Classical Models for Quantum Light

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http://www.mat.univie.ac.at/~neum/papers/physpapers.html#CQlight

Abstract

In this lecture, a timeline is traced from Huygens' wave optics to the modern concept of light according to quantum electrodynamics. The lecture highlights the closeness of classical concepts and quantum concepts to a surprising extent. For example, it is shown that the modern quantum concept of a qubit was already known in 1852 in fully classical terms.

A timeline of light

0: Prehistory

I: Waves or particles? (1679–1801)

II: Polarization (1809–1852)

III: The electromagnetic field (1862–1887)

IV: Shadows of new times coming (1887–1892)

V: Energies are sometimes quantized (1900–1914)

VI: The photon as unitary representation (1926–1946)

VII: Quantum electrodynamics (1948–1949)

VIII: Coherence and photodetection (1955–1964)

IX: Stochastic and nonclassical light (1964-2004)

I picked for each period only one particular theme.

A timeline of light 0: Prehistory

In the beginning God created the heavens and the earth.
And God said, "Let there be light", and there was light.
(Genesis 1:1.3)

It took him only 10 seconds...
10s after the Big Bang: photon epoch begins

2134 BC: oldest solar eclipse recorded in human history
(2 Chinese astronomers lost their head for failing to predict it)

ca. 1000 BC: *It is the glory of God to conceal things,*
but the kings' pride is to research them.
(King Solomon, Proverbs 25:2)

212 BC: Archimedes of Syracuse burns war ships
(using mirrors reflecting sunlight)

A timeline of light I: Waves or particles?

1679: Rømer – observing Jupiter moons $\rightarrow$ finite speed of light

1690: Huygens – (longitudinal) wave picture of light

1690: Huygens – light can be polarized

1704: Newton – particle picture of light

1801: Young – double-slit interferometer

The tension particle – field is still visible today
(particle physics vs. quantum field theory)

wave equation

$\partial_t^2 u(t, \mathbf{x}) = \partial_{\mathbf{x}}^2 u(t, \mathbf{x})$

momentum representation

$\omega^2 \widetilde{u}(\omega, \mathbf{p}) = \mathbf{p}^2 u(\omega, \mathbf{p})$ $(\hbar = 1, c = 1)$

dispersion relation

$\omega^2 = \mathbf{p}^2$

Characteristic rays perpendicular to wave fronts represent light rays (geometric optics)

A timeline of light II: Polarization

1809: Malus – polarization law; pure states of single qubit

1818: Fresnel – diffraction: synthesis of Huygens and Young

1821: Fresnel – polarization: transverse wave picture
unpolarized light = transverse waves changing polarization direction very fast

1850: Foucault – speed of light $< c$ in media
disproves particle picture

1852: Stokes – partially polarized light

The transformation behavior of rays of completely polarized light was first described by Etienne-Louis MALUS 1809 (who coined the name "polarization"); that of partially polarized light by George STOKES 1852.

Here I give a modern description of the core of their work.

A ray (quasimonochromatic beam) of polarized light of fixed frequency is characterized by a state, described equivalently by a real **Stokes vector**

$$S = (S_0, S_1, S_2, S_3)$$

with

$$S_0 \geq |\mathbf{S}| = \sqrt{S_1^2 + S_2^2 + S_3^2},$$

or by a **coherence matrix**, a complex positive semidefinite 2×2 matrix ρ. These are related by

$$\rho = \frac{1}{2}(S_0 + \mathbf{S} \cdot \sigma) = \frac{1}{2}\begin{pmatrix} S_0 + S_3 & S_1 - iS_2 \\ S_1 + iS_2 & S_0 - S_3 \end{pmatrix},$$

where σ is the vector of Pauli matrices.

tr $\rho = S_0$ is the **intensity** of the beam.

$p = |\mathbf{S}|/S_0 \in [0, 1]$ is the **degree of polarization.**

A linear, non-mixing (not depolarizing) instrument
(for example a polarizer or phase rotator)
is characterized by a complex 2×2 **Jones matrix** T.

The instrument transforms an in-going beam in the state ρ into an out-going beam in the state $\rho' = T\rho T^*$.
The intensity of a beam after passing the instrument is
$S_0' = \mathrm{tr}\ \rho' = \mathrm{tr}\ T\rho T^* = \mathrm{tr}\ \rho T^* T$.

If the instrument is lossless, the intensities of the in-going and the out-going beam are identical.
This is the case if and only if the Jones matrix T is unitary.

A fully polarized beam (a pure polarization state) has $p = 1$ and is described by a rank 1 polarization matrix, which can be written in the form $\rho = \psi\psi^*$ with a state vector ψ determined up to a phase.

In this case, the intensity of the beam is

$$S_0 = \langle 1 \rangle = |\psi|^2 = \psi^*\psi.$$

A **polarizer** has $T = \phi\phi^*$, where $|\phi|^2 = 1$.
It reduces the intensity to

$$S_0' = \langle T^*T \rangle = |\phi^*\psi|^2.$$

This is **Malus' law**.

An instrument with Jones matrix T transforms a beam in the pure state ψ into a beam in the pure state $\psi' = T\psi$.

Passage through inhomogeneous media can be modeled by means of many slices consisting of very thin instruments with Jones matrices $T(t)$ close to the identity.

For small Δt (the time needed to pass through one slice) we may assume that

$$T(t) = 1 - \frac{i\Delta t}{\hbar} H(t) + O(\Delta t^2).$$

In the lossless case $T(t)$ is unitary, hence $H(t)$ is Hermitian.

If $\psi(t)$ denotes the pure state at time t then $\psi(t+\Delta t) = T(t)\psi(t)$, so that

$$i\hbar\frac{d}{dt}\psi(t) = \frac{i\hbar}{\Delta t}(\psi(t+\Delta t)-\psi(t))+O(\Delta t^2) = \frac{i\hbar}{\Delta t}(T(t)-1)\psi(t)+O(\Delta t^2).$$

In a continuum limit $\Delta t \to 0$ we obtain

$$T(t) = 1 - \frac{i\Delta t}{\hbar}H(t) + O(\Delta t^2),$$

giving the time-dependent **Schrödinger equation**

$$i\hbar\frac{d}{dt}\psi(t) = H(t)\psi(t).$$

Schrödinger was not yet born in 1852.

But we see that a polarized quasimonochromatic beam of classical light behaves exactly like a modern quantum bit.

A linear, mixing (depolarizing) instrument transforms ρ instead into a sum of several terms of the form $T\rho T^*$.

It is therefore described by a superoperator,
a real 4×4 **Mueller matrix** acting on the Stokes vector.

Equivalently, it is described by a completely positive
linear map on the space of 2×2 matrices,
acting on the polarization matrix.

STOKES' 1852 paper already contains all the modern quantum phenomena for qubits, explained in classical terms:

Splitting polarized beams into two beams with different, but orthogonal polarization corresponds to writing a wave functions as superposition of preferred basis vectors.

Mixtures are defined (in his paragraph 9) as arising from "groups of independent polarized streams" and give rise to partially polarized beams.

The coherence matrix is represented by Stokes
with four real parameters, in today's terms comprising
the Stokes vector.

Stokes asserts (in his paragraph 16) the impossibility
of recovering from a mixture of several distinct pure states
any information about these states beyond what is encoded
in the Stokes vector (equivalently, the polarization matrix).

The latter can be linearly decomposed in many
essentially distinct ways into a sum of pure states,
but all these decompositions are optically indistinguishable.

A timeline of light III: The electromagnetic field

1862: Maxwell – light is a form of electromagnetic radiation

1887: Hertz – Maxwell equations match all known properties of light

At this point, the classical theory is in some sense consolidateted. But as we shall see later, not all has been said yet.

In vacuum, where light travels with a constant speed c, its **mode** is characterized by an arbitrary nonzero complex vector potential $\mathbf{A}(\mathbf{x}, t)$ satisfying the free **Maxwell equations**.

In the radiation gauge, these take the form

$$c^{-2}\partial_t^2\mathbf{A}(\mathbf{x}, t) = \Delta\mathbf{A}(\mathbf{x}, t), \quad \nabla \cdot \mathbf{A}(\mathbf{x}, t) = 0,$$

The **energy density** of light,

$$\rho(\mathbf{x}, t) := |\mathbf{A}(\mathbf{x}, t)|^2,$$

integrates over space to the total energy at time t.

A timeline of light IV: Shadows of new times coming

Once the classical understanding of light was sound, new effects were discovered that lead to quantum mechanics.

1887: Hertz – photoelectric effect

1887: Michelson & Morley – speed of light independent of moving observer

1892: Poincaré – Poincaré sphere $S_1^2 + S_2^2 + S_3^2 = S_0^2$

We'll encounter the Poincaré sphere again in quantum mechanics....

A timeline of light V: Energies are sometimes quantized

1900: Lenard – UV light ionizes gases

1900: Planck – black body radiation explained, $E = h\nu = \hbar\omega$

1905: Einstein – photoelectric effect via discrete quanta of light

1907: Silberstein – complex vector representation for light

1910 Bateman – conformal invariance of Maxwell equations

1914: Millikan – confirmed Einstein's law.

Electric and magnetic field

$$E(t, \mathbf{x}) = \partial_t A(t, \mathbf{x}), \quad B(t, \mathbf{x}) = \nabla \times A(t, \mathbf{x})$$

Silberstein vectors

$$\psi(t)(\mathbf{x}) := E(t, \mathbf{x}) + iB(t, \mathbf{x})$$

Fourier transform

$$\psi(\mathbf{p}) := E(0, \mathbf{p}) + iB(0, \mathbf{p}))$$

transversality

$$\mathbf{p} \cdot \psi(\mathbf{p}) = 0$$

dispersion relation

$$E = |\mathbf{p}|c, \quad \omega(\mathbf{p}) = |\mathbf{p}|c/\hbar$$

A timeline of light VI: The photon as unitary representation

1926: Schrödinger – quantum mechanical wave equation

1926: Lewis – photons (wrong picture, name stuck)

1926: Wentzel – photoelectric effect

1927: Dirac – semirelativistic multiparticle QED

1928: Wiener – coherence matrices and quantum theory

1929 Soleillet – Mueller matrices for linear optical elements

1931: Oppenheimer – photon wave function (Silberstein vector)

1939: Wigner – unitary irreps of Poincaré group

1941: Jones – linear optical elements for polarized light

1946: Bloch – Bloch sphere for spin 1/2

Polarization was recognized as a quantum phenomenon only 78 years after Stokes, when quantum mechanics was already fully developped.

In 1928, Norbert WIENER exhibited a description in terms of the Pauli matrices and wrote:

"It is the conviction of the author that this analogy" between classical optics and quantum mechanics "is not merely an accident, but is due to a deep-lying connection between the two theories".

Wiener showed that the description of polarization in terms of the Poincare sphere (1892) is identical to the description of 2-level systems (qubits) in terms of what is later called the Bloch sphere (1946).

In modern terminology, polarization is a manifestation
of the massless spin 1 nature of the
irreducible unitary representation of the Poincare group
defining photons.

A single photon has essentially the same degrees of freedom
as a classical radiation field.

Thus its modes are again described by the free Maxwell equations.
The Hilbert space of **photon wave functions** is the space
of divergence-free vector fields in the spatial region Ω of interest,
corresponding to possible photon modes restricted to a fixed time.

For $\Omega = \mathbb{R}^3$, the correct inner product (which leads to a spin 1 representation of the Poincaré group or conformal group) is given by

$$\langle \mathbf{A}|\mathbf{A}'\rangle_t = \frac{1}{2\pi^2\hbar c}\int \frac{d\mathbf{x}d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}(\nabla\times\mathbf{A}(\mathbf{x},t))^*(\nabla\times\mathbf{A}'(\mathbf{y},t)).$$

For more general Ω, there are additional boundary terms, and the normalization constant changes.

For $\Omega = \mathbb{R}^3$ (only), the momentum representation (Fourier transform) has simpler properties.

Classical electromagnetic radiation
$= 1$ photon in a pure quantum state

Modes $\mathbf{A}$ satisfying these equations and differing by a space-independent phase factor describe the same single photon state $|\mathbf{A}\rangle$.

Hilbert space
$\mathbb{H} = \{\psi \in L^2(\mathbb{R}^3) \mid \mathbf{p} \cdot \psi(\mathbf{p}) = 0\}$
expressing the zero mass and the transversality of photons.

Hamiltonian
$H = \hbar\omega(\mathbf{p})$

inner product
$\langle\phi|\psi\rangle := \int d\mathbf{p}\overline{\phi(\mathbf{p})}\psi(\mathbf{p})$

time dependence
$\psi(t)(\mathbf{p}) = E(t, \mathbf{p}) + iB(t, \mathbf{p}) = e^{-it\omega(\mathbf{p})}\psi(\mathbf{p})$

A timeline of light VII: Quantum electrodynamics

1948: Feynman, Tomonaga, Schwinger – QED radiative corrections

1949: Newton & Wigner – position operators

According to quantum electrodynamics (QED), the most accurately verified theory in physics, the photon is a single-particle excitation of the free quantum electromagnetic field, more specifically, an eigenstate of the photon number operator with eigenvalue 1.

Similarly, an eigenstate of the photon number operator with eigenvalue N describes an N-photon state – "containing" (in a somewhat vague way) N simultaneous photons by a wave function in the N-fold symmetric tensor product of the 1-photon space.

Coherent states, on the other hand, have an indefinite particle number and are characterized by a mode, again a solution of the free Maxwell equations.

But first we take a closer look at the notion of a photon.

To be able to talk consistently about the probability
of a photon being in a particular subregion of Ω,
one needs a family of commuting projection operators
which restricts a mode to such a subregion.

The projection operators are defined as multiplication
with the corresponding characteristic function.

These come for particles such as the electron from
the spectral decomposition of a **position operator**.
To reflect Euclidean position, the components of this operator
must commute and transform like a 3-vector.

But it is impossible to find for photons a vector-valued position operator with commuting components which transforms under rotations like a vector.

This impossibility has been observed without proof by NEWTON & WIGNER in 1949, and was proved in 1962 by WIGHTMAN. The key argument is that multiplying a photon mode $\mathbf{A}(\mathbf{x}, t)$ by a space-dependent scalar function does not result in a photon mode since photons are transverse. Thus the projection operators lead out of the Hilbert space.

An excellent review about failed attempts to circumvent this conclusion was given by BIALYNICKI-BIRULA in `http://lanl.arxiv.org/abs/quant-ph/0508202`

This implies that **there is no consistent notion of probability density for photons**.

Thus the textbook interpretation of Schrödinger wavefunctions is **not applicable** to photons.

The expression analogous to the probability density of a Schrödinger particle,

$$\rho(\mathbf{x}, t) := |\mathbf{A}(\mathbf{x}, t)|^2,$$

integrates over space to the total mean energy of the photon at time t.

Hence $\rho(\mathbf{x}, t)$ is the **mean energy density** of the photon.

Thus it is impossible to talk consistently about the probability for a photon being located in a particular subregion of Ω.

However, one can talk of the energy density of a photon in any such subregion.

This rules out the interpretation of photons as point particles; **photons are intrinsically nonlocal objects**.

On the other hand, photons prepared in a laboratory are located there, hence have an approximate position.

Thus there must exist a weaker form of localization.

The only natural conclusion:

Photons must be regarded as localized lumps of energy
in a beam of light.
This view allows one to represent a sequence of photons as multiple temporally sepated lumps of energy in the same time-variant beam.

The shape of the lump is encoded in the mode of the photon.
This makes photons **divisible** objects,
not all-one things like point particles.
Upon emission by a whole atom or molecule, whole photons are created.

The indivisibility is instead in the irreducibility of the representation!

A timeline of light VIII: Coherence and photodetection

1955: Wolf – mutual coherence function

1957: Fano – entangled photons (positronium annihilation)

1963: Glauber – coherent photon states for laser light

1964: Mandel, Sudarshan & Wolf – photoeffect without photons

WOLF: coherence functions for arbitrary e/m radiation
(discussed later)

FANO: 4×4 coherence matrices for unpolarized 2-photon states

GLAUBER: The modes of coherent light
(described by **coherent states**)
are in 1-1 correspondence with the solutions of
the free Maxwell equations.

We concentrate on the 1964 item on the photoelectric effect

MANDEL, SUDARSHAN & WOLF showed
(in far more detail than WENTZEL 1926 before) that
photodetection of laser light can be correctly explained with
(deterministic) classical fields and (noisy) quantum detectors.

Indeed, for normal light and for laser light, most quantum effects
are reproducible with classical electromagnetic fields
(STROUD & JAYNES 1970).

Quantum field theory is needed only
to explain the nonclassical correlations observed
when experimenting with nonclassical light.
(Negative coefficients in the P-representation)

See, e.g., CLAUSER 1974 for the limitations of
classical fields in case of coincidence measurements.

In their 2008 survey “The concept of the photon”, Muthukrishnan et al. write: “In fact, it can be shown that the essence of **the photoelectric effect does not require the quantization of the radiation field,** a misconception perpetuated by the mills of textbooks”.

Some people, such as the Nobel prize winner Willis Lamb 1995 even take this as an indicator that *”there is no such thing as a photon”*.

Thus, although a quantum effect, the photoelectric effect has nothing direct to do with quantum properties of light.

A rigorous treatment of the photoelectric effect for classical light
requires the solution of the Dirac equation for an electron
in a periodic potential representing a crystal,
with a quasimonochromatic external electromagnetic field.

Usually only a simplified treatment is given.

Immediately after Heisenberg and Schrödinger
discovered the modern form of quantum mechanics,
two distinct quantum treatments of the photoelectric effect
were given in 1926 by Wentzel and in 1927 by Dirac.

Dirac used "light quanta", now called photons,
after Lewis 1926. (But Lewis' photons were
supposed to be conserved, while real photons are not.)

Wentzel's derivation was instead based on
the response of a **quantum detector**
to a **classical** electromagnetic field.
A much more thorough version of his treatment
was given by Mandel, Sudarshan & Wolf 1964.

Fearn & Lamb (1991) treat the bare bones of the
photoelectric effect with classical light in a simpler,
exactly solvable toy approximation.
The electrons are reduced to a 1-dimensional quantum system
in a δ-function potential, and light is treated as a
time-dependent external field contribution to the Hamiltonian.

As today's standard semiclassical treatment
(with a classical treatment of light
and a quantum treatment of electrons)
we may consider the treatment in the
quantum optics book by MANDEL & WOLF 1995.

The main observation is that both the quantum and
the classical treatment of the photoelectric effect
produce detection rates
(rather than a **single** detection probability)
as the response to a quasimonochromatic classical mode
(or quantum coherent state in that mode)
of the electromagnetic field.

The rate of clicks is proportional to the intensity of the beam.

The electrons in all models show a response correctly reproducing the behavior of a photodetector fed with coherent light.

The so-called P-representation of SUDARSHAN extends this result for coherent states to a large class of other states encountered in practice, including normal (thermal) light. (Nonclassical light where the correspondence fails exists, but must be prepared by special means.)

This shows conclusively that **not photons but electrons**, the only quantum mechanical system in these models, are responsible for the quantum effects in the photodetector.

The **probabilistic results** of low intensity photodetection are therefore the consequence of the interaction with the ensemble of highly localized particles in the macroscopic detector:

Myriads of electrons, one or more in each minimum of the periodic potential of the crystal, each serve as a binary detector with a none-or-all response.

The random response is the inevitable response of a random collection of independent one-or-all detection elements to a continuous signal – whether the signal is a classical or a quantum signal.

Probably it does not even matter whether the detection elements are classical or quantum!

Thus we are forced to the conclusion that
there are no random photons
in a quasimonochromatic beam.

This **lack of inherent randomness**
is also the ultimate reason for
why it is possible to produce
deterministically **photons on demand**.

Can a classical treatment be reconciled
with a photon particle picture?

The successful intuitive appeal of photons as particles,
and time and coincidence measurements of photodetection events,
require an explanation.

A frequently heard opinion, popularized by GLAUBER,
”photons are what photodetectors count”,
can be correct only if one also admits the existence of
classical photons in a classical electromagnetic field
– whatever these could be.

A semiclassical picture must necessarily compose the particles out of waves.

The natural solution is to **define a photon as a wave packet**, a localized concentration of energy in a state of the field, with the correct amount $E = \omega\hbar$ of energy.

This definition is independent of the classical or quantum mechanical model, and gives the semiclassical models of the photo effect a natural interpretation.

But some photons (such as those in plane waves) are very delocalized, while others (such as those in high intensity states) are too tightly packed.

In both cases, the field is only artificially separable into photons, quanta of energy $h\nu = \hbar\omega$.

A timeline of light IX: Stochastic and nonclassical light

1964: Mandel; Wolf – classical stochastic Maxwell equations

1977: Kimble, Dagenais & Mandel - photon antibunching

1982: Aspekt – verification of Bell inequalities

2004: Keller et al. – nice example of single photons on demand

The **second order coherence theory** of the
stochastic Maxwell equations gives a valid description
not only for classical beams of light
but for general classical electromagnetic radiation in the vacuum.

The relevant effective observables for classical radiation fields are
the momentum-dependent 3×3 coherence matrices

$$C(\mathbf{p}_1, \mathbf{p}_2, t) := \langle \mathbf{A}(\mathbf{p}_1)\mathbf{A}(\mathbf{p}_2)^* \rangle_t,$$

where $\mathbf{A}(\mathbf{p})$ is the spatial Fourier transform of $\mathbf{A}(\mathbf{x}, 0)$.

If $\mathbf{A}(\mathbf{p})$ is negligible except for momenta $\mathbf{p}$ very close to a fixed
vector, we have a quasimonochromatic beam. Then the coherence
matrices reduce to 3×3 matrices, and in transversal coordinates to
the 2×2 coherence matrices discussed before.

Tomorrow I'll give a second lecture, where
the results of the historical review given today
are utilized to reassess the meaning of probability,
observables and stochastic processes for the
classical and quantum description of light.

In particular we discuss the description of partially coherent,
fluctuating light through classical stochastic Maxwell equations
(with uncertainty in the initial conditions only).

We determine to which extent it describes quantum properties
of light that do not explicitly measure time correlations.

Our last topic today will therefore be an experiment for creating
single photons on demand.

We consider quantum models for photons on demand and their realization through laser-induced emission by a **single** calcium ion in a cavity.

The exposition is based on work at the
Max Planck Institute of Quantum Optics in Garching (Germany).

For details, see the paper

M Keller, B Lange, K Hayasaka, W Lange and H Walther,
A calcium ion in a cavity as a controlled single-photon source,
New Journal of Physics 6 (2004), 95.

from which some figures are taken, and which discusses
an explicit model of their experimental setting.

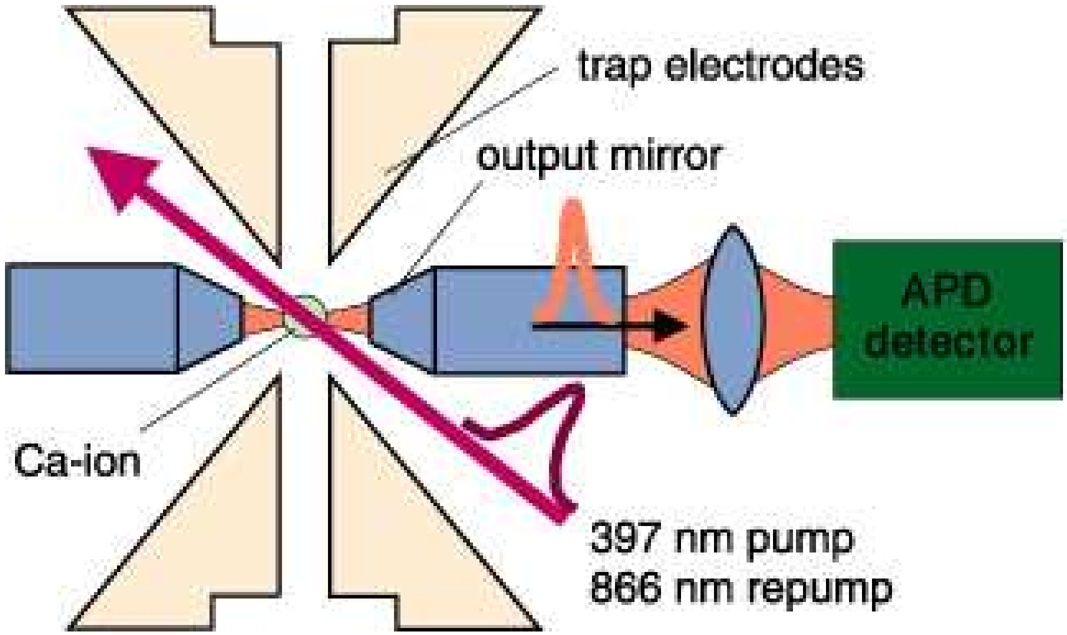


Figure 16. Experimental set-up for the generation of single-photon pulses with an ion-cavity system. The drawing shows a cross-section through the trap, perpendicular to the trap axis.

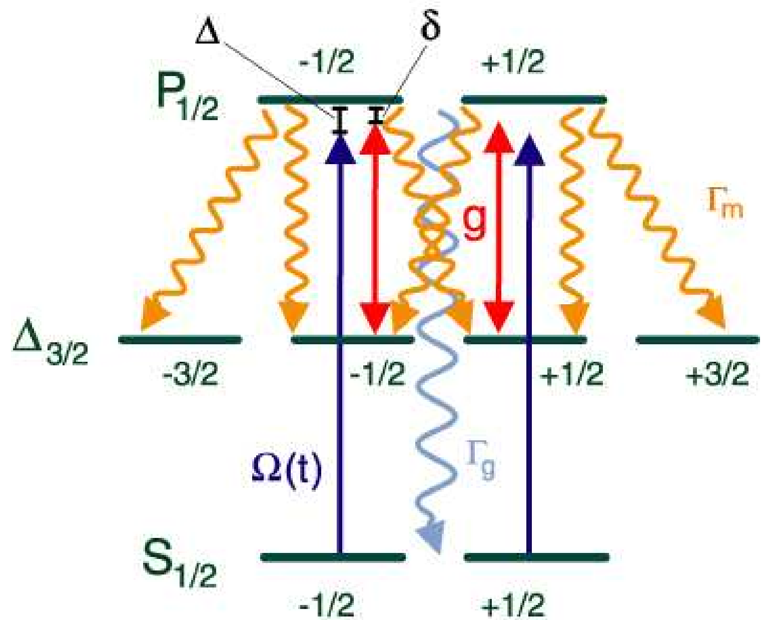
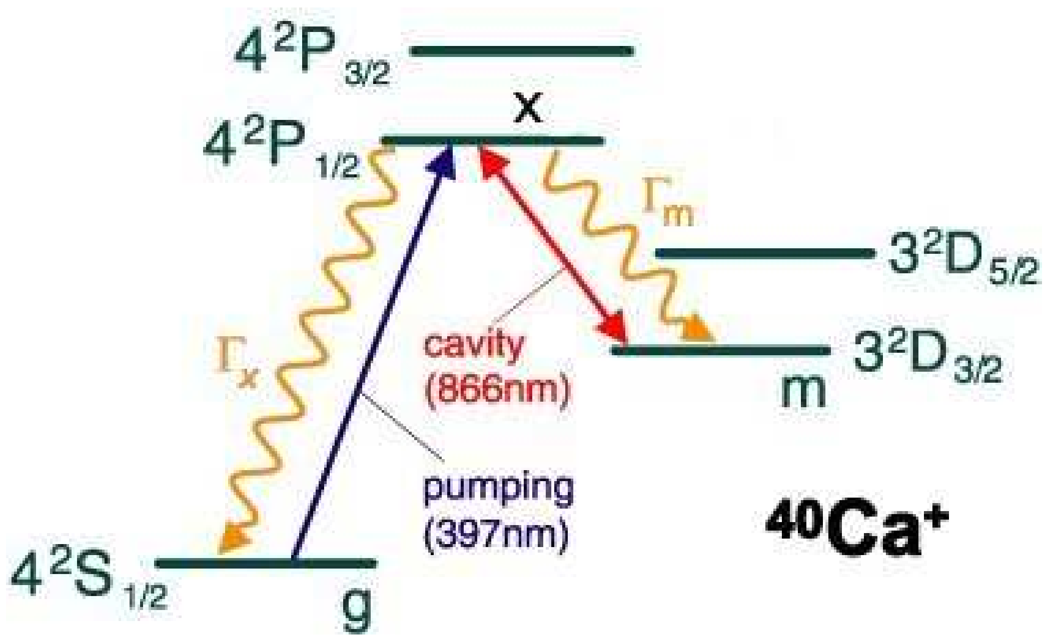
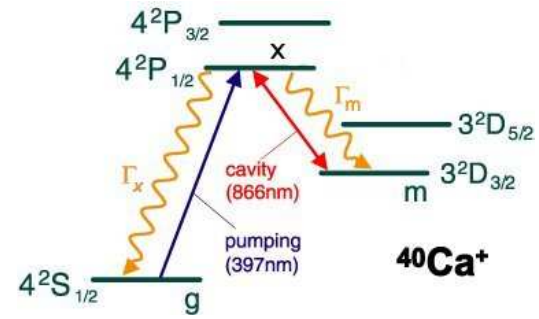


Figure 5. Scheme of the eight-level model on which we base our numerical calculations. Pump and cavity field are assumed to be linearly polarized in the direction of the quantization axis. For clarity, the four possible spontaneous decay transitions to the ground state are represented by a single arrow.

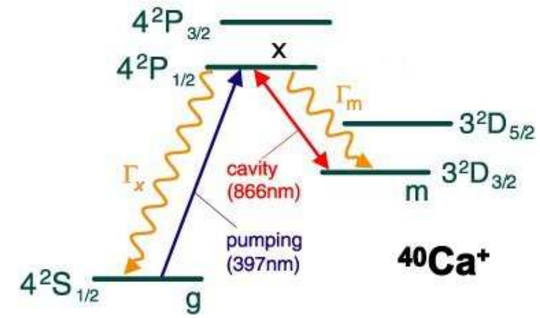
In their paper, KELLER et al. discuss in detail a model based on the following simplified level scheme which ignores the fine structure of the Ca^+ states.

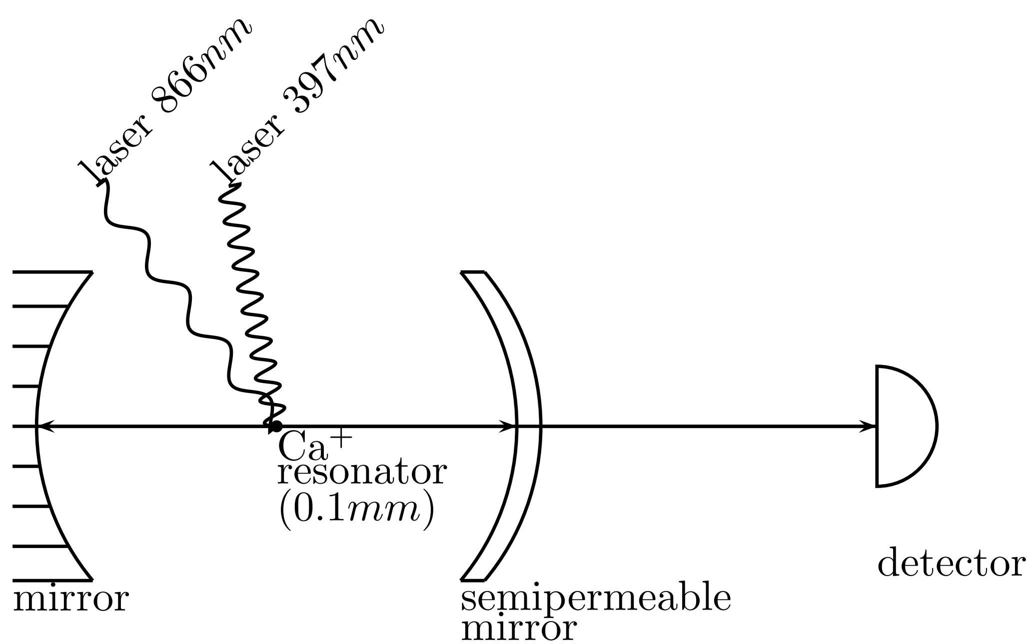




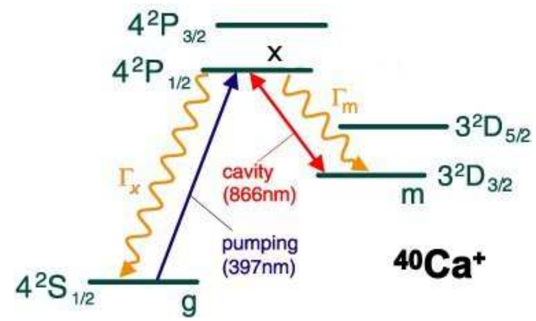
- A **single** ion is localized in the cavity for many hours
- pulsating external fields (lasers) with a total cycle time 100kHz give a predictable rate of single photons
- pump laser at 397nm close to the excitation frequency $S \to P$
- Repeated excitation to P and decay to S until decay into the metastable D state; then inactive
- $\Rightarrow$ produces **exactly** one photon (not counting losses)
- reexcite ion into excited state with a reset laser at 866nm, until it falls back into the ground state

- ground state g, metastable state m, excited state x of Ca^+
- photons $\gamma_{\text{cavity}}, \gamma_{\text{pump}}, \gamma_{\text{reset}}$
- electron e bound in detector





Active processes



- a: $\gamma_{\text{cavity}} \rightleftharpoons \gamma_{\text{cavity}}$ (cavity detuning)
- b: $g + \gamma_{\text{pump}} \rightleftharpoons x$ (excitation)
- c: $x \rightleftharpoons m + \gamma_{\text{cavity}}$ (decay to metastable state)
- d: $\gamma_{\text{cavity}} + e \rightleftharpoons \emptyset$ (photodetection)
- e: $m + \gamma_{\text{reset}} \rightleftharpoons x$ (ion reset)

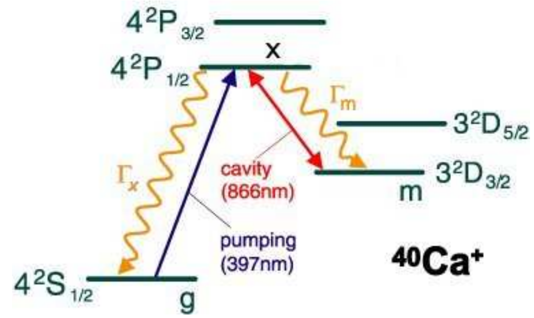
Only a, b, c are modelled explicitly by KELLER et al..

But d, e can be modelled similarly.

Interaction picture model

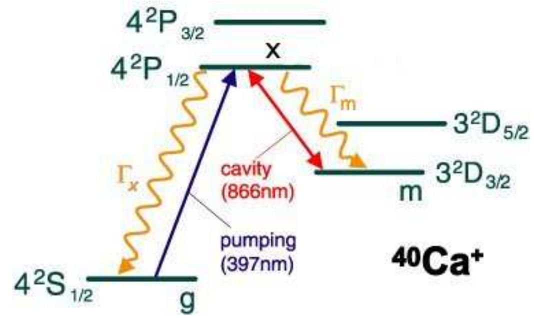
by Keller et al.

(without reset and photodetection)



- a = annihilator of cavity mode of photon
- $b = |g\rangle\langle x|$
- $c = |m\rangle\langle x|$
- Hamiltonian
  $H = \hbar(\delta a^* a + \Delta|g\rangle\langle g| + 2\,\mathrm{Re}\,(\Omega(t)b^* + \mu a c^*))$
- $\delta = \omega_{\text{cavity}} - \omega_{xm}$ cavity detuning
- $\Delta = \omega_{\text{pump}} - \omega_{gx}$ pump detuning
- $\Omega(t)$ classical pulse shape of pump laser
- μ (Keller's g) ion–cavity coupling strength

- $\kappa = 0.02\,\Gamma_x$ cavity loss rate
- $\Gamma_x \approx 138$ MHz (KELLER's Γ_g)
  spontaneous decay rate into ground state



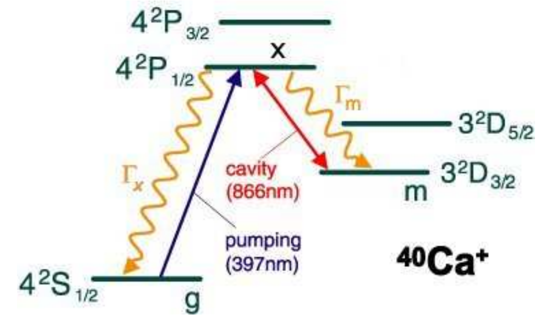
- $\Gamma_m \approx 11$ MHz spontaneous decay rate into metastable state

To account for losses, the dynamics of the density matrix is set up in the form of a

Lindblad master equation

$$\begin{aligned}\dot{\rho} &= -\frac{i}{\hbar}[H,\rho] \quad +\kappa(2a\rho a^* - a^*a\rho - \rho a^*a) \\ &\qquad +\frac{\Gamma_x}{2}(2b\rho b^* - b^*b\rho - \rho b^*b) \\ &\qquad +\frac{\Gamma_m}{2}(2c\rho c^* - c^*c\rho - \rho c^*c)\end{aligned}$$

Note that the master equation is an equation for transition rates; probabilities are obtained by integration over time.



- time-dependent expectations
  $\langle f \rangle_t = \mathrm{tr}\ f\rho(t)$
- time dependent emission rate
  $p(t) = 2\kappa_{\mathrm{tr}} \langle a^* a \rangle_t$
  ($2\kappa_{\mathrm{tr}}$ intensity transmission rate)
- probability of photon emerging from the cavity
  $\eta_{\mathrm{photon}} = \int_0^\infty p(t) dt$
- single-photon efficiency
  $\eta_{\mathrm{abs}} = (\kappa/\kappa_{\mathrm{tr}} - 1)\eta_{\mathrm{photon}}$

The Hilbert space on which the master equation is based is the tensor product of a single mode Fock space for the cavity photon and a 3-mode space for the Ca^+ ion.

An orthonormal basis of the space is given by the kets $|n, k\rangle$, where $n = 0, 1, \ldots$ is the photon occupation number and $k \in \{g, x, m\}$ labels the ion level.

The structure of the Hamiltonian and the dissipation terms in the master equation is such that if the system is started in the ground state $|0, g\rangle$, it evolves to a mixed state in which the photon number is never larger than 1.

Thus multiphoton states do not contribute at all, and one can truncate the cavity photon Fock space to the two modes with occupation number $n = 0, 1$, without changing the essence of the model.

Of interest for the photon production is the projection of the density matrix to the photon space, obtained by tracing over the ion degrees of freedom. This results in an effective time-dependent photon density matrix

$$\rho_{\text{photon}}(t) = \begin{pmatrix} \rho_{00}(t) & \rho_{01}(t) \\ \rho_{10}(t) & \rho_{11}(t) \end{pmatrix},$$

where

$\rho_{11}(t) = p(t)$ is the probability density of finding a photon,

$\rho_{00}(t)$ is the probability density of finding no photon, and

$\rho_{01}(t) = \rho_{10}(t)^*$ measures the amount of entanglement between the 1-photon state and the vacuum state.

Semidefiniteness of the state requires $|\rho_{01}| \le \sqrt{p(1-p)}$.

Assuming for simplicity that we have approximate equality,
ρ_{photon} is essentially rank one,

$$\rho_{\mathrm{photon}}(t) \approx \psi(t)\psi(t)^*, \quad \psi(t) = s(t)|0\rangle + c(t)|1\rangle,$$

where $s(t)$ and $c(t)$ are functions with $|s(t)|^2 + |c(t)|^2 = 1$,
determined only up to a time-dependent phase factor.
In particular, we may take $c(t)$ to be real and nonnegative.

Thus, in the approximation considered,
the quantum electromagnetic field is in a superposition
of the vacuum mode and the single-photon field mode,
with a 1-photon amplitude $c(t) = \sqrt{p(t)}$
that varies with time and encodes the
probability density $p(t)$ of detecting a photon particle.

In the actual experiments, $p(t)$ has a bell-shaped form,
and the total photon detection probability,
referred to as the **efficiency**, is significant,
but smaller than 1.

Discarding the vacuum contribution corresponding
to the dark, unexcited cavity, and giving up
the interaction picture by inserting the field description
$|1\rangle_t = e^{-i\omega t}\psi_0(\mathbf{x})$ of the photon mode,
the (now time-dependent) 1-photon state takes the form

$$\mathbf{A}_{1\text{photon}}(\mathbf{x}, t) = \sqrt{p(t)}e^{-i\omega t}\psi_0(\mathbf{x}).$$

At this stage one notices a minor discrepancy
with the field description, since the 1-photon state is
no longer an exact solution of the Maxwell equations.

To correct this deviation from Maxwell's equations, one has to work with quasimonochromatic modes and the paraxial approximation.

This results in a more accurate time-dependent 1-photon state, describing a bell shaped electromagnetic field pulse $\mathbf{A}(x,t)$ which solves the Maxwell equation exactly.

How does one to talk in QED about a sequence of N single photons in a laser beam?

If we use a reset mechanism to create a periodic sequence
of excitation-reset cycles of the ion in the cavity,
we get a pulsed stream of many photons.
As before, we find that the electromagnetic field corresponding
to the sequence of pulses is a single, periodically excited
1-photon mode of the electromagnetic field.

Thus (at low intensity), what appears at the photodetector
as a **sequence** of photon particles arriving
is from the perspective of quantum electrodynamics
the manifestation of a **single** nonstationary,
pulsed 1-photon state of the electromagnetic field!

Related material

Recent publications and preprints

`http://www.mat.univie.ac.at/~neum/papers/physpapers.html`

Arnold Neumaier and Dennis Westra,

Classical and Quantum Mechanics via Lie algebras, 2008, 2011.

`http://lanl.arxiv.org/abs/0810.1019`

A **theoretical physics FAQ**

`http://www.mat.univie.ac.at/~neum/physfaq/physics-faq.html`

with topics such as:

What is the meaning of probabilities?

Postulates for the formal core of quantum mechanics

What is a photon?

Physics Overflow

A question and answer site for graduate+ level physics

`http://www.physicsoverflow.org`

Thank you for your attention!