Invitation to Coherent Spaces

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The topological and functional analytic aspects are joint work with Arash Ghaani Farashahi (John Hopkins University, Baltimore)

For details on the mathematics see
`http://www.mat.univie.ac.at/~neum/cohSpaces.html`

The notion of a **coherent space** is a nonlinear version of the notion of a complex Hilbert space:

The vector space axioms are dropped while the notion of inner product, now called a **coherent product**, is kept.

Every coherent space Z can be embedded into a Hilbert space, called the completed **quantum space** of Z, by suitably extending the coherent product to an inner product.

Coherent spaces provide a setting for the study of geometry in a different direction than traditional metric, topological, and differential geometry.

Just as it pays to study the properties of manifolds independently of their embedding into a Euclidean space, so it appears fruitfulto study the properties of coherent spaces independent of their embedding into a Hilbert space.

Coherent spaces have close relations to reproducing kernel Hilbert spaces, Fock spaces, and unitary group representations – and to many other fields of mathematics, statistics, and physics.

The completed quantum spaces of coherent spaces may be viewed as ”reproducing kernel Hilbert spaces without measures”.

The origin of coherent spaces

Let us begin at the origins....

In the beginning God created the heavens and the earth.
And God said, "Let there be light", and there was light.
(Genesis 1:1.3)

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As we now know, it took him only 10 seconds:

10s after the Big Bang, the photon epoch begins.

Today, photons are modelled in terms of **coherent states** (SCHRÖDINGER 1926) – directly related to the present lecture.

For us mortals, understanding the Creation is a fascinating but much slower process. For me, part of the fascination is the discovery of the mathematical structure of physics, that may be viewed as part of God's thoughts when he designed the universe.

Even almost a century after Schrödinger, our understanding of quantum mechanics is still limited, with important unsolved problems in

- the foundations of quantum mechanics (measurement problem),
- the foundations of quantum field theory (**does QED exist?**),

and

- quantum gravity (what is it, really?)

The last two problems can be phrased as **construction problems for certain infinite-dimensional coherent manifolds**.

Coherent spaces and coherent manifolds

Coherent manifolds may be viewed as a new, geometric way of working with concrete Hilbert spaces in which they are embeddable.

In place of measures and integration, differentiation is the basic tool for evaluating inner products.

This makes many calculations easy that are difficult in Hilbert spaces whose inner product is defined through an integral.

One of the strengths of the coherent space approach is that it makes many different things look alike.

Coherent spaces have close relations to many important fields of mathematics, statistics, and physics.

Coherent spaces combine the rich, often highly characteristic variety of symmetries of traditional geometric structures with the computational tractability of traditional tools from numerical analysis and statistics.

In particular, there are relations to

(i) Christoffel–Darboux kernels for orthogonal polynomials,
(ii) Euclidean representations of finite geometries,
(iii) zonal spherical functions on symmetric spaces,
(iv) coherent states for Lie groups acting on homogeneous spaces,
(v) unitary representations of groups,
(vi) abstract harmonic analysis,
(vii) states of C^*-algebras in functional analysis,
(viii) reproducing kernel Hilbert spaces in complex analysis,
(ix) Pick–Nevanlinna interpolation theory,
(x) transfer functions in control theory,
(xi) positive definite kernels for radial basis functions,
(xii) positive definite kernels in data mining,

and

(xiii) positive definite functions in probability theory,
(xvi) exponential families in probability theory and statistics,
(xv) the theory of random matrices,
(xvi) Hida distributions for white noise analysis,
(xvii) Kähler manifolds and geometric quantization,
(xviii) coherent states in quantum mechanics,
(xix) squeezed states in quantum optics,
(xx) inverse scattering in quantum mechanics,
(xxi) Hartree–Fock equations in quantum chemistry,
(xxii) mean field calculations in statistical mechanics,
(xxiii) path integrals in quantum mechanics,
(xxiv) functional integrals in quantum field theory,
(xxv) integrable quantum systems.

After all this advertisement, let us proceed to the mathematics itself!

Definitions

A **Euclidean space** is a complex vector space $\mathbb{H}$ with a binary operation that assigns to $\phi, \psi \in \mathbb{H}$ the **Hermitian inner product** $\phi^*\psi \in \mathbb{C}$, antilinear in the first and linear in the second argument, such that

$$\overline{\phi^*\psi} = \psi^*\phi, \tag{1}$$

$$\psi^*\psi > 0 \quad \text{for all } \psi \in \mathbb{H} \setminus \{0\}. \tag{2}$$

$\mathbb{H}$ has a natural locally convex topology in which the inner product is continuous, and is naturally embedded into its antidual $\mathbb{H}^\times$, the space of continuous antilinear functionals on $\mathbb{H}$. The Hilbert space completion $\overline{\mathbb{H}}$ sits between these two spaces,

$$\mathbb{H} \subseteq \overline{\mathbb{H}} \subseteq \mathbb{H}^\times.$$

A **coherent space** is a nonempty set Z with a distinguished function $K : Z \times Z \to \mathbb{C}$, called the **coherent product**, such that

$$\overline{K(z,z')} = K(z',z), \tag{3}$$

and for all $z_1, \ldots, z_n \in Z$, the $n \times n$ matrix G with entries $G_{jk} = K(z_j, z_k)$ is positive semidefinite.

The **distance** (PARTHASARATHY & SCHMIDT 1972)

$$d(z,z') := \sqrt{K(z,z) + K(z',z') - 2\,\mathrm{Re}\ K(z,z')}, \tag{4}$$

of two points $z, z' \in Z$ is nonnegative and satisfies the triangle inequality.

The distance is a metric precisely when the coherent space is **nondegenerate**, i.e., iff

$$K(z'', z') = K(z, z') \qquad \text{for all} \;\; z' \in Z \Rightarrow \qquad z'' = z.$$

In the resulting topology, the coherent product is continuous.

A **coherent manifold** is a smooth ($= C^\infty$) real manifold Z with a smooth coherent product $K : Z \times Z \to \mathbb{C}$ with which Z is a coherent space. In a nondegenerate coherent manifold, the infinitesimal distance equips the manifold with a canonical Riemannian metric.

A **quantum space** $\mathbb{Q}(Z)$ of Z is a Euclidean space spanned (algebraically) by a distinguished set of vectors $|z\rangle$ $(z \in Z)$ called **coherent states** satisfying

$$\langle z|z'\rangle \;=\; K(z,z') \quad \text{for } z,z' \in Z \tag{5}$$

with the linear functionals

$$\langle z| := |z\rangle^*$$

acting on $\mathbb{Q}(Z)$.

Coherent states with distinct labels are distinct iff Z is nondegenerate.

A construction of Aronszajn 1943 (attributed by him to Moore), usually phrased in terms of reproducing kernel Hilbert spaces, proves the following basic result.

Moore–Aronszajn Theorem:
Every coherent space has a quantum space.
It is unique up to isometry.

The antidual $\mathbb{Q}^\times(Z) := \mathbb{Q}(Z)^\times$ of the quantum space $\mathbb{Q}(Z)$ is called the **augmented quantum space**. It contains the **completed quantum space** $\overline{\mathbb{Q}}(Z)$, the Hilbert space completion of $\mathbb{Q}(Z)$,

$$\mathbb{Q}(Z) \subseteq \overline{\mathbb{Q}}(Z) \subseteq \mathbb{Q}^\times(Z).$$

Examples

Example

Any subset Z of a Euclidean space is a coherent space with coherent product $K(z, z') := z^*z'$.

Conversely, any coherent space arises in this way from its quantum space.

Example: Klauder spaces

The Klauder space $KL[V]$ over the Euclidean space V is the coherent manifold $Z = \mathbb{C} \times V$ of pairs $z := [z_0, \mathbf{z}] \in \mathbb{C} \times V$ with coherent product $K(z, z') := e^{\overline{z}_0 + z'_0 + \mathbf{z}^* \mathbf{z}'}$.

The quantum spaces of Klauder spaces are essentially the **Fock spaces** introduced by Fock 1932 in the context of **quantum field theory**; they were first presented by Segal 1960 in a form equivalent to the above.

$KL[\mathbb{C}]$ is essentially in Klauder 1963. Its coherent states are precisely the scalar multiples of the coherent states discovered by Schrödinger 1926.

The quantum space of $KL[\mathbb{C}^n]$ was systematically studied by Bargmann 1963.

Example: The Bloch sphere

The unit sphere in $\mathbb{C}^2$ is a coherent manifold Z_{2j+1} with coherent product $K(z, z') := (z^* z')^{2j}$ for some $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

It corresponds to the Poincaré sphere (or Bloch sphere) representing a single quantum mode of an atom with spin j, or for $j = 1$ the polarization of a single photon mode.

The corresponding quantum space has dimension $2j + 1$. The associated coherent states are the so-called **spin coherent states**.

In the limit $j \to \infty$, the unit sphere turns into the coherent space of a classical spin, with coherent product

$$K(z, z') := \begin{cases} 1 & \text{if } z' = \overline{z}, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting quantum space is infinite-dimensional and describes **classical stochastic motion** on the Bloch sphere in the Koopman representation.

This example shows that the same set Z may carry many interesting coherent products, resulting in different coherent spaces with nonisomorphic quantum spaces.

Example: The classical limit

More generally, any coherent space Z gives rise to an infinite family of coherent spaces Z_n on the same set Z but with modified coherent product $K_n(z, z') := K(z, z')^n$ with a nonnegative integer n.

The quantum space $\mathbb{Q}(Z_n)$ is the symmetric tensor product of n copies of the quantum space $\mathbb{Q}(Z)$.

If Z has a physical interpretation and the classical limit $n \to \infty$ exists, it usually has a physical meaning, too. In this case, it is equivalent to the limit where **Planck's constant** $\hbar$ goes to zero.

Example: Quantum spaces of entire functions

A **de Branges function** (DE BRANGES 1968) is an entire analytic function $E : \mathbb{C} \to \mathbb{C}$ satisfying

$$|E(\overline{z})| < |E(z)| \quad \text{if } \operatorname{Im} z > 0.$$

With the coherent product

$$K(z, z') := \begin{cases} \overline{E}'(\overline{z})E(z') - E'(\overline{z})\overline{E}(z') & \text{if } z' = \overline{z}, \\ \dfrac{\overline{E}(\overline{z})E(z') - E(\overline{z})\overline{E}(z')}{2i(\overline{z} - z')} & \text{otherwise,} \end{cases}$$

$Z = \mathbb{C}$ is a coherent space.

The corresponding quantum spaces are the de Brange spaces relevant in complex analysis.

Symmetry

Let Z be a coherent space. A map $A : Z \to Z$ is called **coherent** if there is an **adjoint map** $A^* : Z \to Z$ such that

$$K(z, Az') = K(A^*z, z') \quad \text{for } z, z' \in Z \tag{6}$$

If Z is nondegenerate, the adjoint is unique, but not in general.

A **symmetry** of Z is an invertible coherent map on Z with an invertible adjoint.

Coherent maps form a semigroup $\operatorname{Coh} Z$ with identity; the symmetries from a group.

An **isometry** is a coherent map A that has an adjoint satisfying $A^*A = 1$. An invertible isometry is called **unitary**.

Example: Distance regular graphs

The orbits of groups of linear self-mappings of a Euclidean space define coherent spaces with predefined transitive symmetry groups.

For example, the symmetric group Sym(5) acts as a group of Euclidean isometries on the 12 points of the icosahedron in $\mathbb{R}^3$.

The coherent space consisting of these 12 points with the induced coherent product therefore has Sym(5) as a group of unitary symmetries.

The quantum space is $\mathbb{C}^3$.

The skeleton of the icosahedron is a distance-regular graph, i.e., the number of neighbors of a vertex is constant, and the number of neighbors of a vertex y closer to or further away from another vertex x depends only on the distance of x and y in the graph.

Many more interesting examples of finite coherent spaces are related to Euclidean representations of distance-regular graphs and other highly symmetric combinatorial objects.

Quantization

Quantization I

Let Z be a coherent space and $\mathbb{Q}(Z)$ a quantum space of Z. Then for any coherent map A on Z, there is a unique linear map $\Gamma(A) : \mathbb{Q}(Z) \to \mathbb{Q}^\times(Z)$ such that

$$\Gamma(A)|z\rangle = |Az\rangle \quad \text{for all } z \in Z. \tag{7}$$

We call $\Gamma(A)$ the **quantization** of A and Γ the **quantization map**.

The quantization map furnishes a representation of the semigroup of coherent maps on Z on the quantum space of Z.

In particular, this gives a **unitary representation** of the group of unitary coherent maps on Z.

For a coherent manifold, the symmetry group is usually a Lie group generated by first order differential operators called **coherent differential operators**.

Under weak conditions, every coherent differential operator X can be quantized by promoting it to the self-adjoint operator

$$d\Gamma(X) := \frac{d}{dt}\Gamma(e^{\iota tX}\Big|_{t=0}.$$

In the quantum physics applications, $\iota = i/\hbar$.

The **infinitesimal quantization map** $d\Gamma$ has natural properties; for example, it is linear and the relation $e^{\iota t d\Gamma(X)} = \Gamma(e^{\iota tX})$ holds.

Example: Möbius space

The **Möbius space** $Z = \{z \in \mathbb{C}^2 \mid |z_1| > |z_2|\}$ is a coherent manifold with coherent product $K(z, z') := (\overline{z}_1 z_1' - \overline{z}_2 z_2')^{-1}$.

A quantum space is the Hardy space of analytic functions on the complex upper half plane with Lebesgue integrable limit on the real line.

The Möbius space has a large semigroup of coherent maps (a semigroup of compressions, Olshanski 1981) consisting of the matrices $A \in \mathbb{C}^{2\times 2}$ such that

$$\alpha := |A_{11}|^2 - |A_{21}|^2, \quad \beta := \overline{A}_{11} A_{12} - \overline{A}_{21} A_{22}, \quad \gamma := |A_{22}|^2 - |A_{12}|^2$$

satisfy the inequalities

$$\alpha > 0, \quad |\beta| \le \alpha, \quad \gamma \le \alpha - 2|\beta|.$$

It contains as a group of symmetries the group $GU(1,1)$ of matrices preserving the Hermitian form $|z_1|^2 - |z_2|^2$ up to a positive factor.

This example generalizes to central extensions of all **semisimple Lie groups** and associated **symmetric spaces** or **symmetric cones** and their **line bundles**.

These provide many interesting examples of coherent manifolds.

This follows from work on coherent states discussed in monographs by PERELOMOV 1986, by FARAUT & KORÁNYI 1994, by NEEB 2000.

In particular, the nonclassical states of light in quantum optics called **squeezed states** are described by coherent spaces corresponding to the **metaplectic group**; cf. related work by NERETIN 2011.

An **involutive coherent manifold** is a coherent manifold Z equipped with a smooth mapping that assigns to every $z \in Z$ a **conjugate** $\bar{z} \in Z$ such that $\overline{\overline{z}} = z$ and $\overline{K(z, z')} = K(\overline{z}, \overline{z}')$ for $z, z' \in Z$.

Under additional conditions, an involutive coherent manifold carries a canonical **Kähler structure**; the (possibly multi-valued) **Kähler potential** is given by the logarithm of the coherent product.

The coherent quantization of these manifolds is equivalent to traditional **geometric quantization of Kähler manifolds**.

In the coherent setting, quantization is not restricted to finite-dimensional manifolds. Thus it extends **geometric quantization** to (certain) infinite-dimensional manifolds.

Quantization II

A function $m : Z \to \mathbb{C}$ is called **homogeneous** iff

$$|z'\rangle = \lambda|z\rangle \Rightarrow \quad m(z')|z'\rangle = \lambda m(z)|z\rangle, \tag{8}$$

for all $\lambda \in \mathbb{C}$ and $z, z' \in Z$.

A kernel $X : Z \times Z \to \mathbb{C}$ is called **homogeneous** if, for all $z \in Z$, the functions $X(\cdot, z), X(z, \cdot)$ are homogeneous.

For every homogeneous kernel X there is a unique linear operator $N(X)$ from $\mathbb{Q}(Z)$ to its algebraic antidual, called the **normal ordering** of X, such that

$$\langle z|N(X)|z'\rangle = X(z,z')K(z,z') \quad \text{for } z,z' \in Z. \tag{9}$$

Equivalently, $N(X)$ defines a sesquilinear form on $\mathbb{Q}(Z)$.

We call a kernel $X : Z \times Z \to \mathbb{C}$ **normal** if $N(X)$ maps $\mathbb{Q}(Z)$ to $\mathbb{Q}^\times(Z)$.

If the coherent product vanishes nowhere then any linear operator $\mathbf{F} : \mathbb{Q}(Z) \to \mathbb{Q}(Z)^\times$ is the normal ordering of a unique normal, homogeneous kernel X.

The normal ordering generalizes the corresponding notion in **Fock spaces**, i.e., quantum spaces of Klauder spaces, to the quantum spaces of arbitrary coherent spaces.

Quantization III

We call a coherent space Z **slender** if any finite set of linearly dependent, nonzero coherent states in a quantum space $\mathbb{Q}(Z)$ of Z contains two parallel coherent states.

Slenderness proofs are quite nontrivial and exploit very specific properties of the coherent states of the coherent spaces.

Examples of interesting slender coherent spaces are the **Möbius space** and the **Klauder spaces**.

Let Z be a slender coherent space and let $\mathbb{Q}(Z)$ be a quantum space of Z.

Then for every homogeneous function $m : Z \to \mathbb{C}$ there is a unique linear operator $\mathbf{a}(m) : \mathbb{Q}(Z) \to \mathbb{Q}(Z)$ such that

$$\mathbf{a}(m)|z\rangle = m(z)|z\rangle \quad \text{for } z \in Z. \tag{10}$$

The $\mathbf{a}(m)$ generalize the **annihilation operators** in Fock spaces to the quantum spaces of arbitrary slender coherent spaces.

Klauder spaces and Fock spaces

Klauder spaces have a large semigroup of coherent maps, which contains a large unitary subgroup. The **oscillator semigroup** over V is the semigroup $Os[V]$ of matrices

$$A = [\rho, p, q, \mathbf{A}] := \begin{pmatrix} 1 & p^* & \rho \\ 0 & \mathbf{A} & q \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Lin}(\mathbb{C} \times V \times \mathbb{C})$$

with $\rho \in \mathbb{C}$, $p \in V^\times$, $q \in V$, and $\mathbf{A} \in \mathrm{Lin}(V, V^\times)$; one easily verifies the formulas for the product

$$[\rho, p, q, \mathbf{A}][\rho', p', q', \mathbf{A}'] = [\rho' + \rho + p^* q', \mathbf{A}'^* p + p', q + \mathbf{A}q', \mathbf{A}\mathbf{A}']$$

and the identity $1 = [0, 0, 0, 1]$. With the adjoint

$$[\rho, p, q, \mathbf{A}]^* = [\overline{\rho}, q, p, \mathbf{A}^*],$$

$Os[V]$ is a *-semigroup consisting of coherent maps acting on $[z_0, \mathbf{z}] \in Kl[V]$ as

$$[\rho, p, q, \mathbf{A}][z_0, \mathbf{z}] := [\rho + z_0 + p^* \mathbf{z}, q + \mathbf{A}\mathbf{z}]. \tag{11}$$

The subset of coherent maps of the form

$$W_\lambda(q) := [\tfrac{1}{2}(q^*q + i\lambda), -q^*, q, 1] \quad (q \in V, \lambda \in \mathbb{R})$$

is the **Heisenberg group** $H(V)$ over V. We have

$$W_\lambda(q)W_{\lambda'}(q') = W_{\lambda+\lambda'+\sigma(q,q')}(q + q'),$$

$$W_\lambda(q)[z_0, \mathbf{z}] = [\tfrac{1}{2}(q^*q + \lambda) + z_0 - q^*z, q + \mathbf{z}]$$

with the symplectic form

$$\sigma(q, q') = 2\,\mathrm{Im}\, q^*q'. \tag{12}$$

The n-dimensional **Weyl group** is the subgroup of $H(\mathbb{C}^n)$ consisting of the $W_\lambda(q)$ with real λ and q.

In Klauder spaces, homogeneous maps and kernels are independent of z_0 and z_0'. We introduce an abstract **lowering symbol** a and its formal adjoint, the abstract **raising symbol** a^*.

For $f : V \to \mathbb{C}$, the map $\widetilde{f} : Z \to \mathbb{C}$ with $\widetilde{f}(z) := f(\mathbf{z})$ is homogeneous. hence there are linear operators

$$f(a) := \mathbf{a}(\widetilde{f}), \quad f(a^*) := \mathbf{a}^*(\widetilde{f}).$$

In particular, if $V = \mathbb{C}^n$ we define

$$a_k := f_k(a), \quad a_k^* := f_k(a^*),$$

where f_k maps $\mathbf{z}$ to $\mathbf{z}_k$. Thus formally, a is a column vector whose n components are the **lowering operators** a_k, also called **annihilation operators**. Similarly, a^* is a row vector whose n components are the **raising operators** a_k, also called **creation operators**.

For $F : V \times V \to \mathbb{C}$, we define the homogeneous kernel $\widetilde{F} : Z \times Z \to \mathbb{C}$ with $\widetilde{F}(z, z') := F(\mathbf{z}, \mathbf{z}')$.

We have the conventional notation

$$:F(a^*, a): \; := N(\widetilde{F})$$

for **normally ordered expressions** in creation and annihilation operators, interpreted as linear operators from $\mathbb{Q}(Z)$ to its algebraic dual.

The **renormalization** issues in quantum field theory are related to the problem of ensuring that normally ordered expressions are defined as continuous operators on suitable domains.

Theorem

(i) Every linear operator $A:\mathbb{Q}(Z)\to\mathbb{Q}^\times(Z)$ can be written uniquely in normally ordered form $A = {:}F(a^*,a){:}$.

(ii) The map $F\to {:}F{:}$ is linear, with ${:}1{:}=1$ and

$$:f(a)^*F(a^*,a)g(a): \;=\; f(a)^*\;:F(a^*,a):\;g(a).$$

In particular,

$$:f(a)^*g(a): \;=\; f(a)^*g(a).$$

(iii) The quantized coherent maps satisfy

$$\Gamma(A) \;=\; :e^{\rho+p^*a+a^*q+a^*(\mathbf{A}-1)a}: \quad \text{for } A=[\rho,p,q,\mathbf{A}].$$

(iv) We have the **Weyl relations**

$$e^{p^*a}e^{a^*q} = e^{p^*q}e^{a^*q}e^{p^*a}$$

and the **canonical commutation relations**

$$(p^*a)(a^*q)-(a^*q)(p^*a)=\sigma(p,q),$$

with the symplectic form (12).

Causal coherent manifolds

Causal space-time

A **space-time** is a smooth real manifold M with a Lie group $\mathbb{G}(M)$ of distinguished diffeomorphisms called **space-time symmetries** and a symmetric, irreflexive **causality relation** $\times$ on M preserved by $\mathbb{G}(M)$.

We say that two sections j, k of a vector bundle over M are **causally independent** and write this as $j \times k$ if

$$x \times y \quad \text{for } x \in \operatorname{Supp} j,\ y \in \operatorname{Supp} k.$$

Examples:

Minkowski space-time $M = \mathbb{R}^{1\times d}$ with a Lorentzian inner product of signature $(+^1, -^d)$ and $x \times y$ iff $(x-y)^2 < 0$.
Here d is the number of spatial dimensions; most often $d \in \{1, 3\}$.
$\mathbb{G}(M)$ is the **Poincaré group** $ISO(1, d)$.

Euclidean space-time M with $x \times y$ iff $x \neq y$.
Two Euclidean cases are of particular interest:

For **Euclidean field theory**, $M = \mathbb{R}^4$ and $\mathbb{G}(M)$ is the group $ISO(4)$ of Euclidean motions.

For **chiral conformal field theory**, M is the unit circle and $\mathbb{G}(M)$ is the **Virasoro group**. Its center acts trivially on M but not necessarily on bundles over M.

Coherent spaces for quantum field theory

A **causal coherent manifold** over a space-time M is a coherent manifold Z with the following properties:

(i) The points of Z form a vector space of smooth sections of a vector bundle over M.

(ii) The symmetries in $\mathbb{G}(M)$ act as unitary coherent maps.

(iii) The coherent product satisfies the following causality conditions:

$$K(j,j') = 1 \quad \text{if } j \times j' \text{ or } j \mid\mid j' \tag{13}$$

$$K(j+k,j'+k) = K(j,j') \quad \text{if } j \times k \times j'. \tag{14}$$

Examples:

From any Hermitian quantum field ϕ of a relativistic quantum field theory satisfying the **Wightman axioms**, for which the smeared fields $\phi(j)$ (with suitable smooth real test functions j) are self-adjoint operators, and any associated state $\langle\cdot\rangle$,
the definition

$$K(j,j') := \langle e^{-i\phi(j)}e^{i\phi(j')}\rangle$$

defines a causal coherent manifold.

There are many known classes of relativistic quantum field theories satisfying these properties in 2 and 3 space-time dimensions.

Under additional conditions one can conversely derive from a causal coherent manifold the Wightman axioms for an associated quantum field theory.

In 4 space-time dimensions, only free and quasifree examples satisfying the Wightman axioms are known.

The question of the existence of interacting relativistic quantum field theories in 4 space-time dimensions is completely open.

The future

This is ongoing work in progress.

What I showed you is just the tip of a huge iceberg waiting to be charted and explored....

Thank you for your attention!

The topological and functional analytic aspects are joint work with Arash Ghaani Farashahi (John Hopkins University, Baltimore)

For the discussion of questions concerning coherent spaces, please use the discussion forum

`https://www.physicsoverflow.org`

For a copy of the slides, and for details on the mathematics (at present only partially written up), see

`http://www.mat.univie.ac.at/~neum/cohSpaces.html`