Explicit effective Hamiltonians for general linear quantum-optical networks

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Abstract

Linear optical networks are devices that turn classical incident modes by a linear transformation into outgoing ones. In general, the quantum version of such transformations may mix annihilation and creation operators. We derive a simple formula for the effective Hamiltonian of a general linear quantum network, if such a Hamiltonian exists. Otherwise we show how the scattering matrix of the network is decomposed into a product of three matrices that can be generated by Hamiltonians.

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Simple optical instruments [1] such as beam splitters or parametric amplifiers are characterized by linear input-output relations. The beam splitter transforms the annihilation operator of the incident light modes according to the classical laws of optical interference, i.e., by a linear transformation. The parametric amplifier acts like a phase-conjugating mirror, combining the annihilation operator of one incident mode with the creation operator of the other. Complex optical networks can be constructed from beam-splitters, mirrors and active elements such as parametric amplifiers [2, 3, 4].

Networks are essential to the optical processing of quantum information. Furthermore, linear networks may possess interesting quantum-statistical properties when they are large, for instance as representations of the Ising model [6, 5] and as examples of quantum localization [7]. Here we derive a simple formula for the Hamiltonian of an arbitrarily large quantum-optical network, if such a Hamiltonian exists. Otherwise we show how the scattering matrix of the network is decomposed into a product of three matrices that can be generated by Hamiltonians. Our results allows to predict how the quantum state of the incident light modes is processed. Our theory contains as special cases the previously studied Hamiltonians of symmetric passive networks [4], the theory of the beam splitter [8, 9, 10] and of the parametric amplifier [11, 12].

Consider n incident modes with the annihilation operators $\widehat{a}_k$ and the creation operators $\widehat{a}_k^\dagger$, the index k running from 1 to n. Suppose that the optical network produces n' outgoing modes with the corresponding operators $\widehat{a}_k'$ and $\widehat{a}_k'^\dagger$. (As we shall see in (7) below, n' must equal n.)

Canonical quantization of the linear input-output behavior of a classical linear network leads to the requirement that the mode operators are related to each other by the linear transformation

$$\begin{pmatrix} \widehat{a}_k' \\ \widehat{a}_k'^\dagger \end{pmatrix} = \mathbf{S} \begin{pmatrix} \widehat{a}_k \\ \widehat{a}_k^\dagger \end{pmatrix} . \tag{1}$$

(In this formula, $\widehat{a}_k$ is representative for the vector formed by all annihilation operators, and similar for the other operators indexed by k.) $\mathbf{S}$ denotes the classical scattering matrix of the network; it has $2n'$ rows and $2n$ columns and must be of the form

$$\mathbf{S} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} , \tag{2}$$

where $\overline{A}$ denotes the (untransposed) complex conjugate of A. Indeed, (2) is necessary for the consistency of (1), as can be seen by writing in components and forming the adjoint. If $B = 0$, the network is passive and can be constructed from beam splitters and mirrors [2]. Otherwise, the transformation (1) mixes annihilation and creation operators, and we speak of an active network.

In the Schrödinger picture, incident light in the state $|\psi\rangle$ is transformed into the state $|\psi'\rangle = \widehat{S}|\psi\rangle$ of the outgoing light, with a unitary scattering operator $\widehat{S}$. In the Heisenberg picture, this corresponds to the transformation $\widehat{O}' = \widehat{S}^\dagger \widehat{O} \widehat{S}$ for arbitrary operators $\widehat{O}$. In particular, to match the mode transformation (1), we need to have

$$\mathbf{S}\begin{pmatrix}\widehat{a}_k\\ \widehat{a}_k^\dagger\end{pmatrix}=\widehat{S}^\dagger\begin{pmatrix}\widehat{a}_k\\ \widehat{a}_k^\dagger\end{pmatrix}\widehat{S}. \tag{3}$$

Here it is understood that $\widehat{S}$ acts on each creation and annihilation operator separately. In the following we show how to construct from the classical scattering matrix $\mathbf{S}$ a unitary scattering operator $\widehat{S}$ such that (3) holds; this operator then completely specifies the desired quantum behavior of the network.

The light of both the incident and the outgoing modes consists of bosons subject to the commutation rules

$$\begin{aligned}[][\widehat{a}_k,\widehat{a}_{k'}^\dagger]&=\delta_{kk'}\,, &\qquad [\widehat{a}_k,\widehat{a}_{k'}]&=0\,,\\ [\widehat{a}'_k,\widehat{a}'^\dagger_{k'}]&=\delta_{kk'}\,, &\qquad [\widehat{a}'_k,\widehat{a}'_{k'}]&=0\,.\end{aligned} \tag{4}$$

We write the commutation relations (4) in matrix form,

$$\begin{aligned}\left[\begin{pmatrix}\widehat{a}_k\\ \widehat{a}_k^\dagger\end{pmatrix},(\widehat{a}_{k'}^\dagger,\widehat{a}_{k'})\right]&=\begin{pmatrix}[\widehat{a}_k,\widehat{a}_{k'}^\dagger] & [\widehat{a}_k,\widehat{a}_{k'}]\\ [\widehat{a}_k^\dagger,\widehat{a}_{k'}^\dagger] & [\widehat{a}_k^\dagger,\widehat{a}_{k'}]\end{pmatrix}=\mathbf{G}\,,\\ \left[\begin{pmatrix}\widehat{a}'_k\\ \widehat{a}'^\dagger_k\end{pmatrix},(\widehat{a}'^\dagger_{k'},\widehat{a}'_{k'})\right]&=\begin{pmatrix}[\widehat{a}'_k,\widehat{a}'^\dagger_{k'}] & [\widehat{a}'_k,\widehat{a}'_{k'}]\\ [\widehat{a}'^\dagger_k,\widehat{a}'^\dagger_{k'}] & [\widehat{a}'^\dagger_k,\widehat{a}'_{k'}]\end{pmatrix}=\mathbf{G}'\end{aligned} \tag{5}$$

with

$$\mathbf{G}=\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix},\quad \mathbf{G}'=\begin{pmatrix}\mathbf{1}' & 0\\ 0 & -\mathbf{1}'\end{pmatrix}, \tag{6}$$

where $\mathbf{1}$ and $\mathbf{1}'$ denote the unit matrix in n and n' dimensions, respectively. Inserting the linear mode transformation (1) into the commutation relations and using (2), we find that the commutation relations are equivalent to $\mathbf{G}'=\mathbf{S}\,\mathbf{G}\,\mathbf{S}^\dagger$. This relation implies that $\mathbf{G}\mathbf{S}^\dagger\mathbf{G}'$ is the inverse of $\mathbf{S}$. Since $\mathbf{S}$ is invertible, the number of incident modes must equal the number of outgoing modes,

$$n=n', \tag{7}$$

a result well-known for passive networks made of beam splitters and mirrors [2, 3, 4]. This feature therefore remains true for general quantum-optical networks. In particular, even if just one light beam is transformed into two or more beams a matching number of incident vacuum modes are involved with their vacuum noise affecting the outgoing light.

Since (7) implies $\mathbf{G}'=\mathbf{G}$, we find

$$\mathbf{G}=\mathbf{S}\,\mathbf{G}\,\mathbf{S}^\dagger \tag{8}$$

as another consistency condition for classical scattering matrices. The matrices $\mathbf{S}$ satisfying (8) are called quasi-unitary [13] and form a Lie group $\mathbb{G}_0$ whose infinitesimal generators are the elements of the quasi-unitary Lie algebra $\mathbb{L}_0$ consisting of all matrices $\mathbf{K}$ with

$$\mathbf{K}\mathbf{G}+\mathbf{G}\mathbf{K}^\dagger=0. \tag{9}$$

Because of (6), (9) implies that the elements of $\mathbb{L}_0$ are precisely those of the form

$$\mathbf{K} = \begin{pmatrix} A & D \\ D^\dagger & B \end{pmatrix} \tag{10}$$

with antihermitian A, B and arbitrary D. Because of (2), an $\mathbf{S}$ suitable as a classical scattering matrix in fact belongs to a subgroup $\mathbb{G}$ of $\mathbb{G}_0$; the corresponding Lie subalgebra $\mathbb{L}$ consists of all $\mathbf{K}$ of the form

$$\mathbf{K} = \begin{pmatrix} A & D \\ \overline{D} & \overline{A} \end{pmatrix} \tag{11}$$

with antihermitian A and complex symmetric D.

By standard results for Lie groups, every element $\mathbf{S} \in \mathbb{G}$ can be written (in many ways) as a product

$$\mathbf{S} = e^{\mathbf{K}_1} \cdots e^{\mathbf{K}_m} \tag{12}$$

of finitely many exponentials of infinitesimal generators $\mathbf{K}_1 \ldots \mathbf{K}_m \in \mathbb{L}$.

A special case in which a single exponential often suffices is when $\mathbf{S} \in \mathbb{G}$ can be diagonalized, i.e., $\mathbf{S} = X \Lambda X^{-1}$ with a diagonal matrix Λ, and has no eigenvalues that are real and negative, then $\mathbf{S} = e^{\mathbf{K}}$ with $\mathbf{K} = X(\ln \Lambda) X^{-1}$, where, for the logarithm $\ln \Lambda$, the principal value is taken in each diagonal element. Often, and if $\mathbf{K}$ is sufficiently small, always, $\mathbf{K} \in \mathbb{L}$; then (9) holds.

For general $\mathbf{S} \in \mathbb{G}$, a decomposition (12) of the form

$$\begin{aligned} \mathbf{S} &= \exp \begin{pmatrix} A_1 & 0 \\ 0 & \overline{A_1} \end{pmatrix} \exp \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \exp \begin{pmatrix} A_3 & 0 \\ 0 & \overline{A_3} \end{pmatrix} \\ &= \begin{pmatrix} \exp A_1 & 0 \\ 0 & \overline{\exp A_1} \end{pmatrix} \begin{pmatrix} \cosh D & \sinh D \\ \sinh D & \cosh D \end{pmatrix} \begin{pmatrix} \exp A_3 & 0 \\ 0 & \overline{\exp A_3} \end{pmatrix}, \end{aligned} \tag{13}$$

with antihermitian A_j and real diagonal D can be found constructively (see Appendix A); the resulting $\mathbf{K}_j$ belong to $\mathbb{L}$ since they have the form (11). The factorization can be given a natural physical interpretation, similar to the one of the known factorization of two-mode parametric amplifiers [1, 11, 12]: Any linear network can be thought of acting in three steps. First the incident modes are mixed by a passive network, then they undergo parametric amplification with real squeezing parameters, and finally they are subject to another passive mode transformation. In the special case when the entire network is passive the squeezing parameters are zero.

Given the structure of general group elements $\mathbf{S}$, we may concentrate in the construction of the unitary scattering operator $\widehat{S}$ satisfying (3) on the case of a single exponential $\mathbf{S} = e^{\mathbf{K}}$ with $\mathbf{K} \in \mathbb{L}$. We observe that the operator

$$\widehat{H}_{\mathbf{K}} = \frac{1}{2} (\widehat{a}_k^\dagger, \widehat{a}_k)\, \mathbf{H} \begin{pmatrix} \widehat{a}_k \\ \widehat{a}_k^\dagger \end{pmatrix} \qquad \text{with } \mathbf{H} = -i\mathbf{G}\mathbf{K} \tag{14}$$

is Hermitian. Indeed, $\mathbf{H}$ is Hermitian since

$$\mathbf{H}^\dagger = i\mathbf{K}^\dagger \mathbf{G}^\dagger = i\mathbf{K}^\dagger \mathbf{G} = -i\mathbf{G}\mathbf{K} = \mathbf{H} \tag{15}$$

by (9), and the hermiticity of $\widehat{H}_{\mathbf{K}}$ follows. The operator $\widehat{H}_{\mathbf{K}}$ plays the role of an effective Hamiltonian in a fictitious time evolution for the quantum-optical network [4], generating the linear mode transformation (1) in the Heisenberg picture. (In simple cases, the fictitious time τ corresponds to the fractional depth of the optical device.) To show this we put

$$\widehat{S}(\tau) = \exp(-i\tau\widehat{H}_{\mathbf{K}}) \tag{16}$$

and verify that the operator vectors

$$\widehat{A}(\tau) := e^{\tau\mathbf{K}} \begin{pmatrix} \widehat{a}_k \\ \widehat{a}_k^\dagger \end{pmatrix}, \quad \widetilde{A}(\tau) := \widehat{S}(\tau)^\dagger \begin{pmatrix} \widehat{a}_k \\ \widehat{a}_k^\dagger \end{pmatrix} \widehat{S}(\tau). \tag{17}$$

satisfy the same differential equation

$$\frac{d}{d\tau}\widehat{A}(\tau) = \mathbf{K}\widehat{A}(\tau). \tag{18}$$

Indeed, (18) holds trivially for $\widehat{A}(\tau)$. To show that it also holds for $\widetilde{A}(\tau)$, we use the commutation relations in matrix form (5) and get, using (16),

$$\begin{aligned} \frac{d}{d\tau}\widetilde{A}(\tau) &= i\exp(i\tau\widehat{H}_{\mathbf{K}}) \left[\widehat{H}_{\mathbf{K}}, \begin{pmatrix} \widehat{a}_k \\ \widehat{a}_k^\dagger \end{pmatrix}\right] \exp(-i\tau\widehat{H}_{\mathbf{K}}) \\ &= i\exp(i\tau\widehat{H}_{\mathbf{K}})\,\mathbf{G}\,\mathbf{H} \begin{pmatrix} \widehat{a}_k \\ \widehat{a}_k^\dagger \end{pmatrix} \exp(-i\tau\widehat{H}_{\mathbf{K}}) \\ &= i\mathbf{G}\,\mathbf{H}\,\widehat{S}(\tau)^\dagger \begin{pmatrix} \widehat{a}_k \\ \widehat{a}_k^\dagger \end{pmatrix} \widehat{S}(\tau) \\ &= i\mathbf{G}\mathbf{H}\widetilde{A}(\tau) = \mathbf{K}\widetilde{A}(\tau)\,, \end{aligned} \tag{19}$$

so that the differential equation (18) also holds with $\widetilde{A}(\tau)$ in place of $\widehat{A}(\tau)$. Since $\widehat{A}(\tau)$ and $\widetilde{A}(\tau)$ trivially agree at $\tau = 0$, they agree for all τ. In particular, we conclude that $\widehat{A}(1) = \widetilde{A}(1)$. In view of (16) and (17), this implies that the unitary operator

$$\widehat{S}_{\mathbf{K}} = \exp(-i\widehat{H}_{\mathbf{K}}) \tag{20}$$

satisfies the required equation (3) for $\mathbf{S} = e^{\mathbf{K}}$. Therefore, $\widehat{S}_{\mathbf{K}}$ is the desired scattering operator corresponding to the classical scattering matrix $\mathbf{S} = e^{\mathbf{K}}$.

In the more general case where the classical scattering matrix $\mathbf{S}$ is given by a product (12) of exponentials, one sees by direct substitution that (3) is satisfied by the scattering operator

$$\widehat{S} = \widehat{S}_{\mathbf{K}_m} \cdots \widehat{S}_{\mathbf{K}_1}. \tag{21}$$

Our general formula contains as special cases the known effective Hamiltonians for beam splitters and parametric amplifiers [1, Section 3.3] that are characterized by the scattering matrices

$$\mathbf{S}_{split}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{22}$$

with real ϕ for a beam splitter and

$$\mathbf{S}_{amp}(\zeta) = \begin{pmatrix} \cosh\zeta & 0 & 0 & \sinh\zeta \\ 0 & \cosh\zeta & \sinh\zeta & 0 \\ 0 & \sinh\zeta & \cosh\zeta & 0 \\ \sinh\zeta & 0 & 0 & \cosh\zeta \end{pmatrix} \tag{23}$$

with real ζ for a parametric amplifier. One can easily diagonalize the matrices $\mathbf{S}_{split}(\phi)$ and $\mathbf{S}_{amp}(\zeta)$ and calculate so their logarithms $\mathbf{K}_{split}$ and $\mathbf{K}_{amp}$. However, the product

$$\mathbf{S} = \mathbf{S}_{amp}(\zeta)\mathbf{S}_{split}(\phi) \quad \text{for } \cosh\zeta\cos\phi = 1 \tag{24}$$

does not possess a diagonal representation. This case represents a beam splitter that is exactly compensated by parametric amplification. For example it describes an eavesdropping attempt where quantum information is tapped by beam splitting followed by amplification to restore the light intensity. To find the Hamiltonian for this device, one has to resort to a Jordan decomposition of the scattering matrix. The resulting matrix logarithm is $\mathbf{K} = \mathbf{S} - \mathbf{1}$. Finally, we mention a simple example where the logarithm of the scattering matrix fails to satisfy the Lie condition (9) that is essential for the hermiticity of the Hamiltonian (14). The example is a single-mode squeezer followed by a π phase shifter, characterized by the scattering matrix

$$\mathbf{S} = -\begin{pmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{pmatrix}, \quad \zeta \neq 0. \tag{25}$$

Since the trace of $\mathbf{S}$ is $-2\cosh\zeta < -2$, the matrix $\mathbf{K} = \log\mathbf{S}$ must fail to satisfy (9) for arbitrary choices of the branch of the logarithms. Indeed, if (9) holds, $\mathbf{K}$ has the form (11) with purely imaginary A, hence $\mathbf{K}$ has two real or two purely imaginary eigenvalues $\lambda_{1,2}$. Thus the trace of $S = e^{\mathbf{K}}$ is $e^{\lambda_1} + e^{\lambda_2}$, which cannot be a real number < -2. Consequently, there is no single Hamiltonian that generates a single-mode squeezer followed by a π phase shifter, although both subdevices possess effective Hamiltonians.

To summarize, we have developed an explicit procedure how to calculate the Hamiltonian of a quantum-optical linear network. If the matrix logarithm $\mathbf{K}$ of the scattering matrix $\mathbf{S}$ satisfies the Lie condition (9), the Hamiltonian is given by (14). One can calculate $\mathbf{K}$ by diagonalizing $\mathbf{S}$ or, if this is not possible, using the Jordan decomposition of $\mathbf{S}$. We have shown that the scattering matrix $\mathbf{S}$ of any linear quantum-optical network can be decomposed into three factors that can be diagonalized or are already diagonal. Note that the Hamiltonian (14) is not unique, because the matrix logarithm is multivalued. Furthermore, since any decomposition (12) leads to a realization of the network, there are many equivalent ways to design an optical network with a particular input-output relation. There are also many ways to assemble it from the basic building blocks [2, 3, 4], from beam splitters and parametric amplifiers.

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Appendix A

In this appendix we prove that every $\mathbf{S} \in \mathbb{G}_0$ can be written in the form (12) with three factors of the form

$$\mathbf{K}_2 = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \quad \mathbf{K}_j = \begin{pmatrix} A_{j1} & 0 \\ 0 & A_{j2} \end{pmatrix} \text{ for } j = 1, 3, \tag{26}$$

with antihermitian A_{jk} and real nonnegative diagonal D. In case that $\mathbf{S} \in \mathbb{G}$, the construction can be modified such that (13) hold with antihermitian A_j and real diagonal D.

The proof is constructive and begins by partitioning the matrix $\mathbf{S}$ into four $n \times n$ submatrices,

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}. \tag{27}$$

The condition (8) which expresses the fact that $\mathbf{S} \in \mathbb{G}_0$ implies the equations

$$S_{11}S_{11}^\dagger - S_{12}S_{12}^\dagger = \mathbf{1}, \tag{28}$$

$$S_{11}S_{21}^\dagger = S_{12}S_{22}^\dagger, \tag{29}$$

$$S_{21}S_{21}^\dagger - S_{22}S_{22}^\dagger = -\mathbf{1}. \tag{30}$$

(28) implies that $\|S_{11}^\dagger x\|^2 = x^\dagger S_{11}S_{11}^\dagger x = x^\dagger S_{12}S_{12}^\dagger x + x^\dagger x = \|S_{12}^\dagger x\|^2 + \|x\|^2 > 0$ if $x \neq 0$, and therefore that S_{11} is invertible. It is always possible to factor S_{12} into a product $S_{12} = U_1 S V_2^\dagger$ (singular value decomposition; see, e.g., [14]) consisting of unitary matrices U_1, V_2 and a nonnegative real diagonal matrix S. The matrix $C := (S^2 + 1)^{1/2}$ is a real, nonnegative invertible diagonal matrix commuting with S and satisfying

$$C^2 - S^2 = 1. \tag{31}$$

We now form the matrices

$$V_1 := S_{11}^{-1} U_1 C, \quad U_2 := S_{22} V_2 C^{-1}. \tag{32}$$

This immediately gives $S_{11} = U_1 C V_1^\dagger$ and $S_{22} = U_2 C V_2^\dagger$; moreover (29) implies that $S_{21} = U_2 S V_1^\dagger$. Inserting this into (28) and (30) shows that V_1 and U_2 must be unitary, and insertion into (27) gives

$$\mathbf{S} = \begin{pmatrix} U_1 C V_1^\dagger & U_1 S V_2^\dagger \\ U_2 S V_1^\dagger & U_2 C V_2^\dagger \end{pmatrix} = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} C & S \\ S & C \end{pmatrix} \begin{pmatrix} V_1^\dagger & 0 \\ 0 & V_2^\dagger \end{pmatrix}. \tag{33}$$

Using (8), it is easily seen that each factor in this factorization belongs to $\mathbb{G}_0$. Due to their special form, they can be easily be brought into exponential form. Indeed, since unitary matrices U are normal and have eigenvalues of absolute value one only, they have a spectral factorization $U = Q\exp(i\Phi)Q^\dagger$ with unitary Q and real diagonal Φ, so that $U = e^K$ with antihermitian $K = iQ\Phi Q^\dagger$. Therefore we can find antihermitian A_{jk} such that $U_k = e^{A_{1k}}$ and $V_k^\dagger = e^{A_{3k}}$ for $k = 1, 3$. If we also define the real diagonal matrix $D := \log(C + S)$ with componentwise logarithms (nonnegative since $C \geq 1, S \geq 0$) on the diagonal, it is easily seen that

$$S = \exp\begin{pmatrix} A_{11} & 0 \\ 0 & A_{12} \end{pmatrix} \exp\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} \exp\begin{pmatrix} A_{31} & 0 \\ 0 & A_{32} \end{pmatrix}, \tag{34}$$

with antihermitian A_{jk}, as asserted.

Now suppose that $\mathbf{S} \in \mathbb{G}$. Comparing (2) and (33), we find that we must have

$$\overline{U}_1 C \overline{V}_1^\dagger = U_2 C V_2^\dagger, \quad \overline{U}_1 S \overline{V}_2^\dagger = U_2 S V_1^\dagger. \tag{35}$$

If the diagonal entries of C and S are all distinct, the singular value decomposition is known to be unique up to a diagonal matrix of phases. Therefore, there are diagonal matrices Q_j with $Q_j^\dagger Q_j = 1$ such that

$$\overline{U}_1 = U_2 Q_1, \quad Q_1 \overline{V}_1^\dagger = V_2^\dagger, \quad \overline{U}_1 = U_2 Q_2, \quad Q_2 \overline{V}_2^\dagger = V_1^\dagger. \tag{36}$$

Clearly, this implies that $Q := Q_1 = Q_2$ is real diagonal with $Q^2 = 1$; in particular, Q has diagonal entries ± 1. Therefore

$$\mathbf{S} = \begin{pmatrix} U_1 C V_1^\dagger & U_1 S Q \overline{V}_1^\dagger \\ \overline{U}_1 Q S V_1^\dagger & \overline{U}_1 Q C Q \overline{V}_1^\dagger \end{pmatrix} = \begin{pmatrix} U_1 & 0 \\ 0 & \overline{U}_1 \end{pmatrix} \begin{pmatrix} C & QS \\ QS & C \end{pmatrix} \begin{pmatrix} V_1^\dagger & 0 \\ 0 & \overline{V}_1^\dagger \end{pmatrix}. \tag{37}$$

As before, we can find antihermitian A_j such that $U_1 = e^{A_1}$ and $V_1^\dagger = e^{A_3}$. If we also define the real diagonal matrix $D := \log(C + QS)$ with componentwise logarithms on the diagonal, it is easily seen that (13) holds with antihermitian A_j and real diagonal D.

Finally, if some diagonal entries of C or S coincide, we can perturb $\mathbf{S}$ slightly to remove the degeneracy, and obtain the same decomposition by a limiting argument.

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