

FMATHL, Mathematica, and Wolfram|Alpha

Arnold Neumaier
(University of Vienna, Austria)

Version of January 15, 2010

For the background, see
<http://www.mat.univie.ac.at/~neum/FMathL.html>

Part 1: FMathL

- Why FMathL is needed
- Goals and design criteria
- Who would profit from FMathL?

Part 2: Mathematica and Wolfram|Alpha

- Mathematica features useful for FMathL
- Current limitations of Mathematica and Wolfram|Alpha
- How good answers in Wolfram|Alpha might look like

Part 3: What can FMathL contribute?

- Adding a conceptual level
- How FMathL can benefit Mathematica
- Conclusion

Part 1: FMathL

- Why FMathL is needed
- Goals and design criteria
- Who would profit from FMathL?

Why FMathL is needed

For our consulting work (e.g., electricity market prognosis) in Vienna, we'd like to have a system for efficient mathematical modeling at the highest possible level, namely ...

...in formal mathematical language in an easily readable and easily editable form.

For our work on computer-assisted proofs (of existence of solutions of a PDE, say) we'd like to have a system capable of verifying both the programs that do the numerical part of the proof and the theory that shows that the computations done indeed constitute the desired proof.

Although a number of current systems offer partial capabilities in the desired directions, no existing system satisfies our needs.

The current overhead for using automatic verification software is horrendous:

Freek Wiedijk estimated that it takes 40 human hours to produce a verified version of one page of normal LaTeX, and this work is quite tedious for a mathematician.

But it takes me on average only 4 hours per page to write LaTeX in publication quality, including polishing the presentation style, working on improvements requested by the referees, and correcting galley proofs.

Mathematical markup support in MathML and OpenMath allows the specification of a certain amount of mathematical formalism in a presentation-oriented or content-oriented way.

But neither are the formulas semantically related to the embedding text, nor are these languages sufficiently flexible to handle all formulas common in modern mathematics.

For example, tensor notation is not supported; it is impossible to represent in Content MathML the formula for a covariant derivative using Einstein's summation convention.

Current systems for doing formal mathematics on the computer (automatic theorem provers such as COQ or Theorema) are unaware of the preferences, values, and working habits of mathematicians, with the result that these systems are used by extremely few mathematicians.

But many mathematicians make use of

- mathematical typesetting languages (LaTeX),
- computer algebra packages (e.g., Mathematica),
- high-level numerics languages (e.g., Matlab),
- modeling languages (e.g., AMPL).

Each of these systems has numerous large-scale applications.

This leads to the challenge of designing and building a system (called a mathematical research system, MathResS) that

- is as easy to use as these heavily used systems,
- can encode and check arbitrary mathematics
- feels close to what mathematicians are accustomed to.

Goals and design criteria

In its ideal form, the desired mathematical research system

- represents arbitrary formulas in their natural context;
- represents arbitrary semantical relations between concepts, formulas, names of variables, etc.;
- produces and understands a large part of natural mathematical text;
- the output feels like high quality mathematical prose (whywihyp: "what you write is what you publish");
- feeds arbitrary solvers (packages for computer algebra, partial differential equations, optimization, theorem proving, etc.);
- is context-aware and allows the use of locally consistent notation in its appropriate context even when the notation is globally inconsistent.

Scientific knowledge

provided in form of a database of formal
undergraduate mathematics

- ultimately all mathematics and all sciences –
- makes possible fully formal interactive textbooks
- ultimately proved consistent and correct –
- immediately useful for one's own conceptual work.

A user-extensible database

- seamless integrating the (user-contributed) state
of the art in the field of expertise of the user –
- makes the system very useful
- long before all math/science is encoded.

The system should also be user-configurable (many theories, many personal styles), be incremental, and be capable of learning by doing.

We started work towards these goals, and are still far from having a satisfactory solution for practical applications.

But the partial work we have already done is encouraging enough to believe that the goal is reachable in the near future.

Theoretical investigations

(needed or already begun)

We need to ensure that everything is

- easy to reflect for later automatic verification
- easy to comprehend, both
 - * on the user representation level (interface) and
 - * on the system representation level (for debugging)
- easy to translate between the system level
 - and various dialects of the user level
- easy to extend and maintain,
 - * as easily as ordinary mathematics

This requires research into the formal structure of informal mathematical language,

with strong relations to the foundations of mathematics.

Designing the system also raises interesting philosophical questions, and requires a reconsideration of old ones.

This also requires research into short reflection cycles at various levels:

- axiomatic basis . . . intensional sets
- logic background . . . context logic
- algorithms . . . semantic Turing machines
- language acquisition . . . incremental grammars
- semantics . . . self-explaining representations
- automatic readable proof checking
- readable proof generation

A Mathematician's Dream

What should a smart mathematics editor be able to do?

Whatever we can tell a graduate student to improve in his draft of a paper, in order to make it of publishing quality.

Thus we should be able to give commands or suggestions how the manuscript should change, in a way close to the directives we give to secretaries or graduate students.

For example:

“Adapt the notation to that of this paper”.

Or:

“Find in a mathematical text all variable names denoting integers.

Then replace each one consistently by the smallest of i, j, k, l, m, n that in context does not result in a name conflict, and that consistently names similar uses in a similar way.”

Who would profit from a MathResS?

- Users of mathematics who regularly need to edit mathematical text.
- Users of mathematics who need conceptual support that complements their own expertise.

The appearance of powerful numerical calculators changed the way calculation skills are taught and done.

The appearance of symbolic computation tools such as Mathematica changed the way symbolic calculation skills are taught and done, repeating what happened with calculators on a higher, symbolic level.

Something similar will happen with an FMathL-based MathResS on an even higher, conceptual level.

As mathematics permeates science and engineering, from the most elementary aspects to highly complex modeling tasks, the availability of an FMathL-based system will make it nearly as easy to apply reliably mathematical tools as currently calculators and computer algebra systems are applied.

As our emphasis is on making MathResS able to address large-scale applications, scientists and engineers will directly profit from the ease with which they can do their modeling.

It will no longer be necessary to learn specialized languages for solving mathematical problems – the common mathematical language taught anyway to scientists and engineers will provide direct access to the solution facilities.

As a result,

- modeling cycles will become shorter,
- more complex problems become tractable more easily,
- and experts can concentrate on the parts where their expertise is most needed.

Human expertise will gradually move away from being able to execute repetitive mathematical thinking activities to being able to evaluate the conditions under which such activities are most usefully employed.

Part 2: Mathematica and Wolfram|Alpha

- Mathematica features useful for FMathL
- Current limitations of Mathematica and Wolfram|Alpha
- How good answers in Wolfram|Alpha might look like

Mathematica features useful for FMathL

- elaborate infrastructure
- flexible user interface
- uniform organization principles
- integrated conceptual basis
- declarative, problem-oriented style
- complexity is hidden from the user
- graphs and expressions → semantics
- powerful pattern-matching capabilities
- publishing quality presentations

The original slogan of Mathematica 1.0 in 1986 was “a system for doing mathematics on the computer”.

Now, more than 20 years later, we have Mathematica 7.0, with many impressive mathematical facilities.

But ...

... can the system do mathematics?

Conceptual limitations of Mathematica

Mathematica is a system for computer algebra and numerical calculations, and for presenting results computed on the basis of corresponding algorithms.

Though it has many mathematical capabilities, it is not (and was not designed as) a system for doing general mathematics.

For example, Mathematica 7.0 has no notion of a topological space, a category, or an infinite set.

It also cannot make sense of statements such as “Let H be a Hilbert space and A a densely defined linear operator on H .” or any of its formalized versions, although this could be algorithmic input guiding a proof strategy of a proof system.

Conceptual limitations of Wolfram|Alpha

Wolfram|Alpha is a system for knowledge-based computing.

But it currently (January 15, 2010) has similar limitations on the conceptual level.

There is limited support for logic-based queries, and none for set-theory computations or queries depending on conceptual knowledge.

The query

$$\text{not (not A) = A} \quad (1)$$

results in True, although it is not generally true in intuitionistic logic.

The query

$$A \text{ and not } A \quad (2)$$

does not simplify to False, although this is true even intuitionistically.

The query

$A \text{ union } A$ (3)

is recognized to belong to set theory, but nothing is done with it. The request

simplify $A \text{ union } A$ (4)

is not even understood.

Upon entering

Is Hilbert's 10th problem solvable? (5)

the system says, "Wolfram|Alpha isn't sure how to compute an answer from your input", and suggests to look up Hilbert as a person.

Following the link, one finds indeed the famous mathematician. But no help is given that would enable one to make the original query more understandable to the system.

To the query

Is $L^2(\mathbb{R})$ a Hilbert space? (6)

the system gives the same answer, complemented by links to Hilbert and Hilbert space, something a user asking the query surely knew already about.

Upon entering

category

(7)

the system says, “English word”

but clicking there gives no hint to any
mathematical meaning of the term –

although this is much more likely to have been
intended by someone using a system for
knowledge-based computing.

How good answers might look like

Do not give too much details (one page is usually too much).

But provide descriptive links to what else might be of interest (both to system-generated pages and to external web sites).

Or at least make the verbosity configurable.

Query: $16 \log(2)/(10 \log(3))$

Answer: The result is independent of the basis of the logarithm.

a) Decimal approximation:

1.009487605714331899...

[link to more accuracy](#)

b) Continued fraction: $[1; 105, 2, 2, 59, \dots]$

[link to more accuracy](#)

Related: [link to log](#)

(Series representations and integral representations in some expansion variable as currently given do not make sense here, since no simplification results.)

Query: $a^2 + b^2 = c^2$

Answer:

a) A circular cone in (a, b, c) -space

– links (instead of the full text) to

solve for a variable,

integral solutions,

partial derivatives,

generalization: quadric (conic section)

– related: Fermat's last theorem

b) The theorem of Pythagoras for a right-angled triangle

with sides a, b, c , where c is opposite to the right angle

– links to

proofs of the theorem,

right-angled triangle,

triangle,

Pythagoras from Samos

Query: A and not A

Answer: False in both classical and intuitionistic logic

Related: links to
law of excluded middle
classical logic
intuitionistic logic

Query: Is A and not A true?

Answer: No, in both classical and intuitionistic logic

Related: links to
law of excluded middle
classical logic
intuitionistic logic

Query: $\text{not} (\text{not } A) = A$

Answer:

a) True (in classical logic)

b) undecidable (in intuitionistic logic)

Related: links to
classical logic
intuitionistic logic

Query: Is $\text{not} (\text{not } A) = A$?

Answer:

a) yes in classical logic.

a) in intuitionistic logic only if A is decidable.

Related: links to
double negation law
classical logic
intuitionistic logic

Query: A union A

Answer: A union A = A.

We interpreted your query as one in set theory
(links to alternatives)

Related: link to set theory

Query: simplify A union A

Answer: A

–links to

why?

Clicking on “why?” would provide an intelligible outline of a proof. In more complicated cases, this proof would have embedded more why’s that give more and more detail.

Query: Is $A \cup B = B \cup A$?

Answer: yes.

– links to

why?

Related: links to set theory

Query: Why is $A \cup B = B \cup A$?

Answer: $x \in A \cup B$ iff $(x \in A \text{ or } x \in B)$
iff $(x \in B \text{ or } x \in A)$ iff $x \in B \cup A$.

– links to

extensionality of sets

Query: Is there a standard name for
(A union B) minus (B union A)?

Answer: yes; it is called the symmetric difference
of A and B.

– links to
symmetric difference
set theory

Query: Is there a standard notation for $(A^T A)^{-1} A$?

Answer: yes. If A has full rank, the expression is the (Moore-Penrose) pseudo-inverse, often denoted by A^+ .

– links to

pseudo-inverse

full rank

matrix theory

linear algebra

Query:

How to solve $Ax=b$ when A is symmetric but indefinite?

Answer:

- a) Symmetric indefinite factorization with pivoting
(with link)
- b) link to iterative methods (symmetric matrices)

Related: links to
solving linear equations
linear algebra

Query: Let G be a finite group,
and let H be a subgroup of G .
Given the order of G , are there
restrictions to the order of H ?

Answer: Yes.

The order of H divides the order of G ,
by Cayley's theorem.

– embedded link to Cayley's theorem

Query: Is Hilbert's 10th problem solvable?

Answer: No; it is undecidable.

– links to

Hilbert's 10th problem

Hilbert's problems

Undecidability

Query: Is $L^2(\mathbb{R})$ a Hilbert space?

Answer: Yes.

– links to

$L^2(\mathbb{R})$

Hilbert space

functional analysis

Query: “densely defined linear operator”

Answer: Given a Hilbert space H , a linear operator A is a linear mapping from a subspace $\text{Domain}(A)$ to H ; it is densely defined if $\text{Domain}(A)$ is a dense subspace of H .

Example; In $H = L^2(\mathbb{R})$, the linear operator $A = d/dt$ maps all continuously differentiable functions in H to its derivative. The domain of A consists of all continuously differentiable functions in H and is dense in H , hence A is densely defined.

– links to

Hilbert space

linear operator

dense

domain

Related: links to

self-adjoint

functional analysis

Query: category

Answer:

In mathematics, a category is a mathematical structure consisting of objects and arrows (or morphisms) satisfying certain laws.

More specifically [...]

Link to: alternative interpretations of the query

Part 3: What can FMathL contribute?

- Adding a conceptual level
- How FMathL can benefit Mathematica
- Conclusion

Adding a conceptual level

FMathL would raise Mathematica's and Wolfram|Alpha's capabilities to a more conceptual level, enable users to do or get answered more general mathematics, not only computationally oriented activities.

Users could work on the conceptual level, invoking Mathematica on the fly where needed to answer algorithmic questions, for example checking (or performing) some calculations that are part of an example, an application, or a proof.

Other parts of the proof would be checked by a theorem prover (such as the Mathematica-based Theorema prover).

All standard mathematical concepts could be used, and new concepts could be created and explored both theoretically and algorithmically.

An FMathL-based mathematical research system (MathResS) will provide a natural-language-based semantic framework and a Latex-like interface.

It will allow computers to understand and process arbitrary mathematical definitions, concepts, assertions, and proofs, together with standard undergraduate background knowledge in linear algebra, in real and complex analysis, and a bit beyond these fields.

It will also allow users to expand this knowlege by adding their own expertise to the background theory.

MathResS by itself will provide not primarily algorithms but mathematical understanding – just what Mathematica is currently lacking.

In all current systems with some mathematical capabilities, the understanding resides in the people who created the system (e.g., Mathematica), not in the system itself.

Combined with the tools of computer algebra (as built into Mathematica) and formal logic (as built into systems such as Theorema), the combination MathResS+Mathematica+Theorema will be able to make sense out of general mathematical statements.

It will also be able to place them in the right context, figure out which algorithmic tools are needed to study the topic or question, and which tool to apply under which circumstances.

Some of the proposed answers to the simple queries mentioned above do not need much of FMathL.

But if more complex questions are asked, some understanding of the conceptual mathematical context is required.

If the answer is to be useful for further (and partly formal) work, a semantic representation of the content in the query and the answer is needed – just as Mathematica already has (unlike Latex) a semantic representation of symbolic formulas.

It is here that FMathL shows its full strength.

For questions that are elementary exercises (such as “Prove that every group of exponent 2 is abelian”), it would write answers that read like model solutions.

It could also guide users stepwise to a solution if they are willing to learn for themselves how to solve the exercise.

One could even ask for a pedestrian but straightforward-to-find proof or for a slick, elegant argument.

In more difficult queries the combined system can start a meaningful dialogue with users to find out what they have in mind with their question, and then answer it interactively.

User profiles can be stored and interpreted to find out the most likely context in which to interpret a query.

References to online encyclopedia articles about the subject (or other online references of interest) can be given.

If no straightforward answer is found, the system can try to retrieve scientific papers adapted to the query from scholar.google.com, say.

A combination of Mathematica and Wolfram|Alpha with MathResS will produce answers that are satisfying.

It will produce upon the presentation of yes-no questions not only the right answers yes or no, but also the right reasons why, and perhaps generalizations, etc..

How FMathL can benefit Mathematica

- arbitrary concepts (mathematical and nonmathematical)
- system representation by a semantic matrix (a kind of labelled graph or nonstandard expression)
- user representation:
natural mathematical language,
slightly controlled
- no need to learn a new language
- user-configurable language and presentation

- context recognition
- enter and interpret longer texts
- semantic text manipulation
- scientific text becomes a formal, structured object
- context-sensitive refactoring on the language level
 - style, notation
 - translation, multilingual
 - multiple views of the same material
- conceptual answers can be automatically processed further

- enables one to map the semantic contents of whole areas of mathematics and sciences
- reflection lends the system an understanding of its methods
 - can ask “why?” and get a justification of the answers given
 - automatic verification
 - verified, fully consistent documents
- structural representation of queries and answers
 - enables structural search facilities
 - full semantic control over input and output

Conclusion

One may perhaps rephrase the preceding discussion as follows:

FMATHL will give Wolfram|Alpha and Mathematica the competence of a mathematical advisor rather than just that of a syntactic and computational prodigy.