

Foundations of quantum physics

I. A critique of the tradition

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Abstract. This paper gives a thorough critique of the foundations of quantum physics in its mainstream interpretation (i.e., treating pure states and probability as primitives, without reference to hidden variables, and without modifications of the quantum laws).

This is achieved by cleanly separating a concise version of the (universally accepted) formal core of quantum physics from the (controversial) interpretation issues. The latter are primarily related to measurement, but also to questions of existence and of the meaning of basic concepts like 'state' and 'particle'. The requirements for good foundations of quantum physics are discussed.

Main results:

- Born's rule cannot be valid universally, and must be considered as a scientific law with a restricted domain of validity.
- If the state of every composite quantum system contains all information that can be known about this system, it cannot be a pure state in general.

For the discussion of questions related to this paper, please use the discussion forum <https://www.physicsoverflow.org>.

Contents

1	Introduction	3
2	The traditional foundations of quantum physics	4
2.1	Postulates for the formal core of quantum physics	5
2.2	The pure state idealization	6
2.3	Schrödinger equation and Born's rule	7
2.4	Interpreting the formal core	9
3	A critique of Born's rule	10
3.1	Early, measurement-free formulations of Born's rule	11
3.2	Formulations of Born's rule in terms of measurement	14
3.3	Limitations of Born's rule	16
3.4	The domain of validity of Born's rule	18
3.5	What is a state?	19
3.6	Pure states in quantum field theory	23
4	Requirements for good foundations	24
4.1	Foundations independent of measurement	24
4.2	What is a measurement?	27
4.3	Beables	29
4.4	What is a particle?	32
5	Outlook: New foundations	35
5.1	The thermal interpretation	36
5.2	Questioning the traditional foundations in other respects	36
5.3	Coherent quantum physics	38
	References	40

1 Introduction

This paper gives a thorough critique of the foundations of quantum physics in its mainstream interpretation (i.e., treating pure states and probability as primitives, without reference to hidden variables, and without modifications of the quantum laws).

This is achieved by cleanly separating a concise version of the (universally accepted) formal core of quantum physics (described in Section 2) from the (controversial) interpretation issues. The latter are primarily related to measurement, but also to questions of existence and of the meaning of basic concepts like 'state' and 'particle'.

The bridge between the formal core and measurement is Born's rule, usually assumed to be valid exactly. It is argued in Section 3 that this assumption cannot be maintained, so that Born's rule must be considered as a scientific law with a restricted domain of validity. We also show that if the state of a composite quantum system contains all information that can be known about a system, it cannot be a pure state. These are the main new results of the present paper; they open new possibilities for the foundation of quantum physics.

Section 4 discusses requirements for a foundation of quantum physics free of the major shortcomings of the traditional interpretations. The final Section 5 gives a preview on the alternative foundation discussed in later parts of this series [40, 41, 42, 43], that satisfies these requirements.

To separate the formal core of quantum physics from the interpretation issues we need to avoid some of the traditional quantum mechanical jargon. In particular, following the convention of ALLAHVERDYAN et al. [2], we add the prefix "q-" to all traditional quantum notions that suggest by their name a particular interpretation and hence might confuse the borderline between theory and interpretation. In particular, the Hermitian operators usually called "observables"¹ will be called "q-observables" to distinguish them from observables in the operational sense of numbers obtainable from observation. Similarly, we use at places the terms q-expectation and q-probability for the conventional but formally defined terms expectation and probability.

A number of remarks are addressed to experts and then refer to technical aspects explained in the references given. However, the bulk of this paper is intended to be nontechnical and understandable for a wide audience being familiar with some traditional quantum mechanics. The knowledge of some basic terms from functional analysis is assumed; these are precisely defined in many mathematics books.

In the bibliography, the number(s) after each reference give the page number(s) where it is cited.

¹This notion appears first in DIRAC's 1930 book [16, p.28], where however, Hermiticity is not required. Later editions make this restriction.

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2 The traditional foundations of quantum physics

Quantum physics consists of a **formal core** that is universally agreed upon (basically being a piece of mathematics with a few meager pointers on how to match it with experimental reality) and an interpretational halo that remains highly disputed even after more than 90 years of modern quantum physics. The latter is the subject of the **interpretation of quantum mechanics**, where many interpretations coexist and compete for the attention of those interested in what quantum physics really means.

To set the stage, I give here an axiomatic introduction to the undisputed formal core.

As in any axiomatic setting (necessary for a formal discipline), there are a number of different but equivalent sets of axioms or postulates that can be used to define formal quantum physics. Since they are equivalent, their choice is a matter of convenience.

My choice presented here is the formulation which gives most direct access to statistical mechanics but is free from allusions to measurement. The reason for the first is that statistical mechanics is the main tool for applications of quantum physics to the macroscopic systems we are familiar with. The reason for the second is that real measurements constitute a complex process involving macroscopic detectors, hence should be explained by quantum statistical mechanics rather than be part of the axiomatic foundations themselves. (This is in marked contrast to other foundations, and distinguishes the present system of axioms.)

The relativistic case is outside the scope of the axioms to be presented, as it must be treated by quantum field theory (see the discussion in Part II [40] of this sequence of papers). Thus the axioms described in the following present nonrelativistic quantum statistical mechanics in the Schrödinger picture. (As explained later, the traditional starting point is instead the special case of this setting where all states are assumed to be pure.)

A second reason for my choice is to emphasise the similarity of quantum mechanics and classical mechanics. Indeed, the only difference between classical and quantum mechanics in the axiomatic setting presented below is the following:

- The classical case only works with diagonal operators, where all operations happen point-wise on the diagonal elements. Thus multiplication is commutative, and one can identify operators and functions. In particular, the density operator degenerates into a probability density.
- The quantum case allows for noncommutative operators, hence both q-observables and

the density are (usually infinite-dimensional) matrices.

2.1 Postulates for the formal core of quantum physics

Traditional quantum physics is governed by the following six axioms (A1)–(A6).² Note that all axioms are basis-independent. \hbar is Planck’s constant, and is often set to 1.

(A1) A generic system (e.g., a ‘hydrogen molecule’) is defined by specifying a Hilbert space \mathbb{H} and a densely defined, self-adjoint³ linear operator H called the **Hamiltonian** or the **internal energy**.

(A2) A particular system (e.g., ‘the ion in the ion trap on this particular desk’) is characterized by its **state** $\rho(t)$ at every time $t \in \mathbb{R}$ (the set of real numbers). Here $\rho(t)$ is a Hermitian, positive semidefinite, linear trace class operator on \mathbb{H} satisfying at all times the normalization condition

$$\text{Tr } \rho(t) = 1.$$

Here Tr denotes the trace.

(A3) A system is called **closed** in a time interval $[t_1, t_2]$ if it satisfies the evolution equation

$$\frac{d}{dt}\rho(t) = \frac{i}{\hbar}[\rho(t), H] \quad \text{for } t \in [t_1, t_2], \quad (1)$$

and **open** otherwise. If nothing else is apparent from the context, a system is assumed to be closed.

(A4) Besides the internal energy H , certain other densely defined, self-adjoint operators or vectors of such operators are distinguished as **quantum observables**, short **q-observables**. (E.g., the q-observables for a system of N distinguishable particles conventionally include for each particle several 3-dimensional vectors: the **position** \mathbf{x}^a , **momentum** \mathbf{p}^a , **orbital angular momentum** \mathbf{L}^a and the **spin vector** (or **Bloch vector**) \mathbf{S}^a of the particle with label a . If \mathbf{u} is a 3-vector of unit length then $\mathbf{u} \cdot \mathbf{p}^a$, $\mathbf{u} \cdot \mathbf{L}^a$ and $\mathbf{u} \cdot \mathbf{S}^a$ define the momentum, orbital angular momentum, and spin of particle a in direction \mathbf{u} .)

(A5) For any particular system, and for every vector X of self-adjoint q-observables with commuting components, one associates a time-dependent monotone linear functional $\langle \cdot \rangle_t$ defining the **q-expectation**

$$\langle f(X) \rangle_t := \text{Tr } \rho(t) f(X)$$

²The statements of my axioms contain in parentheses some additional explanations that, strictly speaking, are not part of the axioms but make them more easily intelligible; the list of examples given only has illustrative character and is far from being exhaustive.

³ Self-adjoint operators are Hermitian, $H^* = H$. Hermitian operators have a real spectrum if and only if they are self-adjoint. Hermitian trace class operators are always self-adjoint. The Hille–Yosida theorem says that e^{iX} exists (and is unitary) for a Hermitian operator X if and only if X is self-adjoint; see THIRRING [59] or REED & SIMON [51].

of bounded continuous functions $f(X)$ at time t . (By WHITTLE [65], this is equivalent to a multivariate probability measure $d\mu_t(X)$ on a suitable σ -algebra over the spectrum $\text{Spec } X$ of X defined by

$$\int d\mu_t(X)f(X) := \text{Tr } \rho(t)f(X) = \langle f(X) \rangle_t.$$

This σ -algebra is uniquely determined, and defines **q-probabilities**.)

(A6) Quantum mechanical predictions consist of predicting properties (typically q-expectations or conditional q-probabilities) of the measures defined in Axiom (A5), given reasonable assumptions about the states (e.g., ground state, equilibrium state, etc.)

Axiom (A6) specifies that the formal content of quantum physics is covered exactly by what can be deduced from Axioms (A1)–(A5) without anything else added – except for restrictions defining the specific nature of the states and q-observables, for example specifying commutation or anticommutation relations between some of the distinguished q-observables). Thus Axiom (A6) says that Axioms (A1)–(A5) are complete.

The description of a particular closed system is therefore given by the specification of a particular Hilbert space (in (A1)), the specification of the q-observables (in (A4)), and the specification of conditions singling out a particular class of states (in (A6)). (The description of an open system involves, in addition, the specification of details of the dynamical law.)

Given this, everything predictable in principle about the system is determined by the theory, and hence is predicted by the theory.

At the level of the formal core, q-expectations and all other concepts introduced are only calculational tools that enables one to predict numerical values for theoretical objects with suggestive names. Their precise relation to experimental reality is not specified by the formal core. These relations are the subject matter of the interpretation of quantum mechanics; see Subsection 2.4.

2.2 The pure state idealization

A state ρ is called **pure** at time t if $\rho(t)$ maps the Hilbert space \mathbb{H} to a 1-dimensional subspace, and **mixed** otherwise.

Although much of traditional quantum physics is phrased in terms of pure states, this is a very special case; in most actual experiments the systems are open and the states are mixed states. Pure states are relevant only if they come from the ground state of a Hamiltonian in which the first excited state has a large energy gap. Indeed, assume for simplicity that H has a discrete spectrum. In an orthonormal basis of eigenstates ϕ_k , functions $f(H)$ of

the Hamiltonian H are defined by

$$f(H) = \sum_k f(E_k) \phi_k \phi_k^*$$

whenever the function f is defined on the spectrum. The equilibrium density is the canonical ensemble,

$$\rho(T) = Z(T)^{-1} e^{-H/kT} = Z(T)^{-1} \sum_k e^{-E_k/T} \phi_k \phi_k^*;$$

here k is the **Boltzmann constant**. (Of course, equating this ensemble with equilibrium in a closed system is an additional step beyond our system of axioms, which would require justification.) Since the trace equals 1, we find

$$Z(T) = \sum_k e^{-E_k/kT},$$

the textbook formula for the so-called **partition function**. In the limit $T \rightarrow 0$, all terms $e^{-E_k/T}$ become 0 or 1, with 1 only for the k corresponding to the states with least energy. Thus, if the ground state ϕ_1 is unique,

$$\lim_{T \rightarrow 0} \rho(T) = \phi_1 \phi_1^*.$$

This implies that for low enough temperatures, the equilibrium state is approximately pure. The larger the gap to the second smallest energy level, the better is the approximation at a given nonzero temperature. In particular, the approximation is good if the energy gap exceeds a small multiple of $E^* := kT$.

States of sufficiently simple systems (i.e., those with a few energy levels only) can often be prepared in nearly pure states, by realizing a source governed by a Hamiltonian in which the first excited state has a much larger energy than the ground state. Dissipation then brings the system into equilibrium, and as seen above, the resulting equilibrium state is nearly pure. Those low lying excited states for which a selection rule suppresses the transition to a lower energy state can be made nearly pure in the same way.

2.3 Schrödinger equation and Born's rule

To see how the more traditional setting in terms of the Schrödinger equation arises, we consider the special case of a closed system in a pure state $\rho(t)$ at some time t . The **state vector** of such a system at time t is by definition a unit vector $\psi(t)$ in the range of the pure state $\rho(t)$. It is determined up to a phase factor (of absolute value 1), and one easily verifies that

$$\rho(t) = \psi(t) \psi(t)^*. \tag{2}$$

Remarkably, under the dynamics for a closed system specified in the above axioms, this property persists with time if the system is closed, and the state vector satisfies the **time-dependent Schrödinger equation**

$$i\hbar\dot{\psi}(t) = H\psi(t)$$

Thus the state remains pure at all times. Conversely, for every pure state, the phases of $\psi(t)$ at all times t can be chosen such that the Schrödinger equation holds; the density operator is independent of this phase.

Moreover, if X is a vector of q-observables with commuting components and the spectrum of X is discrete, then the measure from Axiom (A5) is discrete,

$$\int d\mu(X)f(X) = \sum_k p_k f(X_k)$$

with spectral values X_k and nonnegative numbers p_k summing to 1, called **q-probabilities**.⁴ Associated with the p_k are eigenspaces \mathbb{H}_k such that

$$X\psi = X_k\psi \quad \text{for } \psi \in \mathbb{H}_k,$$

and \mathbb{H} is the direct sum of the \mathbb{H}_k . Therefore, every state vector ψ can be uniquely decomposed into a sum

$$\psi = \sum_k \psi_k, \quad \psi_k \in \mathbb{H}_k.$$

ψ_k is called the **projection** of ψ to the eigenspace \mathbb{H}_k . If all eigenvalues of X are discrete and nondegenerate, each \mathbb{H}_k is 1-dimensional and spanned by a normalized eigenvector ϕ_k . Then $X\phi_k = X_k\phi_k$ and the projection is given by $\psi_k = P_k\psi$ with the orthogonal projector $P_k := \phi_k\phi_k^*$, so that

$$\psi = \sum_k \phi_k\phi_k^*\psi.$$

A short calculation using Axiom (A5) now reveals that for a pure state (2), the q-probabilities p_k are given by the **formal Born rule**

$$p_k = |\psi_k(t)|^2 = |\phi_k^*\psi(t)|^2, \tag{3}$$

where $\psi_k(t)$ is the projection of $\psi(t)$ to the eigenspace \mathbb{H}_k .

Identifying these q-probabilities with the probabilities of measurement results (which is already an interpretative step involving (MI) below) constitutes the so-called **Born rule**. Without this identification, the formal Born rule (3) is just a piece of uninterpreted mathematics with suggestive naming.

⁴This leaves open the precise physical meaning of q-probabilities, and the question of how to measure them. This is the price to pay for not entering into interpretational issues.

Deriving the formal Born rule (3) from Axioms (A1)–(A5) makes it feel completely natural, while the traditional approach starting with Born’s rule makes it an irreducible rule full of mystery and only justifiable by its miraculous agreement with certain experiment.

2.4 Interpreting the formal core

In addition to the formal axioms (A1)–(A6), one needs a rudimentary interpretation relating the formal part to experiments. The following **minimal interpretation** seems to be universally accepted.

(MI) Upon measuring at times t_l ($l = 1, \dots, n$) a vector X of q-observables with commuting components, for a large collection of independent identical (particular) systems closed for times $t < t_l$, all in the same state

$$\rho(t_l) = \rho \quad (l = 1, \dots, n)$$

(one calls such systems **identically prepared**), the measurement results are statistically consistent with independent realizations of a random vector X with measure as defined in axiom (A5).

Note that (MI) is no longer a formal statement since it neither defines what **measuring** is, nor what **measurement results** are and what **statistically consistent** or **independent identical system** means. Thus (MI) has no mathematical meaning – it is not an axiom, but already part of the interpretation of formal quantum physics.

(MI) relates the axioms to a nonphysical entity, the social conventions of the community of physicists. The terms ‘measuring’, ‘measurement results’, and ‘statistically consistent’ already have informal meaning in the reality as perceived by a physicist. Everything stated in Axiom (MI) is understandable by every trained physicist. Thus statement (MI) is not an axiom for formal logical reasoning but a bridge to informal reasoning in the traditional cultural setting that defines what a trained physicist understands by reality.

The lack of precision in statement (MI) is on purpose, since it allows the statement to be agreeable to everyone in its vagueness; different philosophical schools can easily fill it with their own understanding of the terms in a way consistent with the remainder.

Interpretational axioms necessarily have this form, since they must assume some unexplained common cultural background for perceiving reality. (This is even true in pure mathematics, since the language stating the axioms must be assumed to be common cultural background.)

(MI) is what *every* traditional interpretation I know of assumes at least implicitly in order to make contact with experiments. Indeed, all traditional interpretations I know of assume much more, but they differ a lot in what they assume beyond (MI).

However, my critique of the universal Born rule in Subsection 3.3 also applies to (MI), since (MI) implies the universal Born rule. Thus (MI) seems to be justified only for certain measurements.

Everything beyond (MI) seems to be controversial; cf. SCHLOSSHAUER cite[Chapter 8]Schl,Schl.book. In particular, already what constitutes a measurement of X is controversial. (E.g., reading a pointer, different readers may get marginally different results. What is the true pointer reading? Does passing a beam splitter or a polarization filter constitute a measurement?)

On the other hand there is an informal consensus on how to perform measurements in practice. Good foundations including a good measurement theory should be able to properly justify this informal consensus by defining additional formal concepts about what constitutes measurement. To be satisfying, these must behave within the theory just as their informal relatives with the same name behave in reality. The goal of the **thermal interpretation** described in Part II [40] and applied to measurement in Part III [41] and Part IV [42] is to provide such foundations. Its link of the formal core to experiment is different from (MI), based instead on the link between quantum physics and thermodynamics known from statistical mechanics.

3 A critique of Born's rule

Traditionally, some version of Born's rule is considered to be an indispensable part of any interpretation of quantum mechanics, either as a postulate, or as a result derived from other postulates, not always on the basis of convincing reasoning. In this section, we have a close look at the possible forms of Born's rule and discuss the limits of its validity.

All traditional foundations of quantum mechanics heavily depend on the concept of (hypothetical, idealized) experiments – far too heavily. This is one of the reasons why these foundations are still unsettled, over 90 years after the discovery of the basic equations for modern quantum mechanics. No other theory has such controversial foundations.

The main reason is that the starting point of the usual interpretations is an idealization of the measurement process that is taken too seriously, namely as the indisputable truth about everything measured. But in reality, this idealization is only a didactical trick for the newcomer to make the formal definitions of quantum mechanics a bit easier to swallow. Except in a few very simple cases, it is too far removed from experimental practice to tell much about real measurement, and hence about how quantum physics is used in real applications.

In experimental physics, measurement is a very complex thing – far more complex than Born's rule (the usual starting point) suggests. To measure the distance between two

galaxies, the mass of the top quark, or the Lamb shift – just to mention three basic examples - cannot be captured by the idealistic measurement concept used there, nor by any of the refinements of it discussed in the literature.

In each of the three cases mentioned, one assembles a lot of auxiliary information and ultimately calculates the measurement result from a best fit of a model to the data. Clearly the theory must already be in place in order to do that. We do not even know what a top quark should be whose mass we are measuring unless we have a theory that tells us this!

The two most accurately determined observables in the history of quantum physics, namely the anomalous magnetic moment of the electron and Lamb shift, are not even q-observables!

To present the stage for the criticism of Born's rule in Subsection 3.3, and related criticism in Subsections 3.4– 3.6, we first need to clarify the meaning of the term "Born's rule". To distinguish different useful meanings we look in Subsections 3.1– 3.2 at the early history of Born's rule.

3.1 Early, measurement-free formulations of Born's rule

It is interesting to consider the genesis of Born's rule,⁵ based on the early papers of the pioneers of quantum mechanics. This and the next subsection benefitted considerably from discussions with Francois Ziegler, though his view of the history is somewhat different (cf. ZIEGLER [73]).

The two 1926 papers by BORN [8, 9] (the first being a summary of the second) introduced the probabilistic interpretation that earned Born the 1954 Nobel prize. Born's 1926 formulation⁶ *"gives the probability for the electron, arriving from the z-direction, to be thrown out into the direction designated by the angles α, β, γ , with the phase change δ "* does not depend on anything being measured, let alone to be assigned a precise numerical measurement value! Instead it sounds like talk about objective properties of electrons ("being thrown out") independent of measurement. Thus Born originally did not relate his interpretation to measurement but to objective properties of scattering processes, no matter whether these were observed.

Rephrased in modern terminology (Born didn't have the concept of an S-matrix), Born's statement above is made precise (and generalized) by the following rule:

⁵apparently named such first in 1934 by BAUER [5, p.302] (*"la règle de Born"*). Before that, during the gestation period of finding the right level of generalization and interpretation, the pioneers talked more vaguely about Born's interpretation of quantum mechanics (or of the wave function). For example, JORDAN [28, p.811] writes about *"Born's Deutung der Lösung [der] Schrödingergleichung"*.

⁶German original, BORN [8, p.865f]: *"bestimmt die Wahrscheinlichkeit dafür, daß das aus der z-Richtung kommende Elektron in die durch α, β, γ bestimmte Richtung (und mit einer Phasenänderung δ) geworfen wird"*

Born's rule (scattering form): In a scattering experiment described by the S-matrix S ,

$$\Pr(\psi_{\text{out}}|\psi_{\text{in}}) := |\psi_{\text{out}}^* S \psi_{\text{in}}|^2$$

is the conditional probability density that scattering of particles prepared in the in-state ψ_{in} results in particles in the out-state ψ_{out} . Here the in- and out-states are asymptotic eigenstates of total momentum, labelled by a maximal collection of independent quantum numbers (including particle momenta and spins).

The scattering form of Born's rule is impeccable and remains until today the basis of the interpretation of S-matrix elements computed from quantum mechanics or quantum field theory.

The 1927 paper by BORN [10] extends this rule on p.173 to probabilities for quantum jumps ("Quantensprung", p.172) between energy eigenstates, given by the absolute squares of inner products of the corresponding eigenstates, still using objective rather than measurement-based language.⁷ For a system initially in state n given by Born's (9), *"the square $|b_{nm}|^2$ is according to our basic hypothesis the probability for the system to be in state m after completion of the interaction"*. Here state n is the n th stationary state (eigenstate with a time-dependent harmonic phase) of the Hamiltonian.

Born derives this rule from two assumptions. The first assumption, made on p.170 and repeated on p.171 after (5), is that an atomic system is always in a definite stationary state.⁸ *"Thus we shall preserve the picture of Bohr that an atomic system is always in a unique stationary state. [...] but in general we shall know in any moment only that, based on the prior history and the physical conditions present, there is a certain probability that the atom is in the n th state."*

Thus for the early Born, the beables of a quantum mechanical system are the quantum numbers of the (proper or improper) stationary states of the system. This assumption works indeed for equilibrium quantum statistical mechanics – where expectations are defined in terms of the partition function and a probability distribution over the stationary states. It also works for nondegenerate quantum scattering theory – where only asymptotic states figure. However, it has problems in the presence of degeneracy, where only the eigenspaces, but not the stationary states themselves, have well-defined quantum numbers. Indeed, [10] assumes – on p.159, remark after (2) and Footnote 2 – that the Hamiltonian has a nondegenerate, discrete spectrum.

⁷*"Das Quadrat $|b_{nm}|^2$ ist gemäß unserer Grundhypothese die Wahrscheinlichkeit dafür, daß das System sich nach Ablauf der Störung im Zustand m befindet"*

⁸*"Wir werden also an dem Bohrschen Bilde festhalten, daß ein atomares System stets nur in einem stationären Zustand ist. [...] im allgemeinen aber werden wir in einem Augenblick nur wissen, daß auf Grund der Vorgeschichte und der bestehenden physikalischen Bedingungen eine gewisse Wahrscheinlichkeit dafür besteht, daß das Atom im n -ten Zustand ist."*

Born's second assumption is his basic hypothesis on p.171 for probabilities for being (objectively) in a stationary state:⁹ *"there is a certain probability that the atom is in the n th state. We now claim that as measure for this probability of state, one must choose the quantity $|c_n|^2 = |\int \psi(x, t)\psi_n^*(x)dx|^2$."*

The 1927 paper by JORDAN [28, p.811] (citing Pauli) extends Born's second assumption further to an objective, measurement independent probability interpretation of inner products (probability amplitudes) of eigenstates of two arbitrary operators, seemingly without being aware of the conceptual problem this objective view poses when applied to noncommuting operators.

The 1927 paper by PAULI [50, p.83, Footnote 1] contains the first formal statement of a probability interpretation for position:¹⁰ *"We shall interpret this function in the spirit of Born's view of the "Gespensterfeld" in [8, 9] as follows: $|\psi(q_1 \dots q_f)|^2 dq_1 \dots dq_f$ is the probability that, in the named quantum state of the system, these coordinates lie simultaneously in the named volume element $dq_1 \dots dq_f$ of position space."* Apart from its objective formulation (no reference to measurement), this is a special case of the universal formulation of Born's rule given below:

The 1927 paper by VON NEUMANN [45, p.45] generalizes this statement to arbitrary selfadjoint operators, again stated as an objective (i.e., measurement independent) interpretation. For discrete energy spectra and their energy levels, we still read p.48: *"unquantized states are impossible" ("nicht gequantelte Zustände sind unmöglich")*.

Note that like Born, Jordan and von Neumann both talk about objective properties of the system independent of measurement. But unlike Born who ties these properties to the stationary state representation in which momentum and energy act diagonally, Pauli ties it to the position representation, where position acts diagonally, and von Neumann allows it for arbitrary systems of commuting selfadjoint operators.

From either Born's or Jordan's statement one can easily obtain the following, basis-independent form of Born's rule, either for functions A of stationary state labels, or for functions A of position:

Born's rule (objective expectation form): The value of a q-observable corresponding to a self-adjoint Hermitian operator A of a system in the pure state ψ (or the mixed state ρ) equals on average the q-expectation value $\langle A \rangle := \psi^* A \psi$ (resp. $\langle A \rangle := \text{Tr } \rho A$).

The first published statement of this kind seems to be in the 1927 paper by LANDAU

⁹ "[...] eine gewisse Wahrscheinlichkeit dafür besteht, daß das Atom im n -ten Zustand ist. Wir behaupten nun, daß als Maß dieser Zustandswahrscheinlichkeit die Größe $|c_n|^2 = |\int \psi(x, t)\psi_n^*(x)dx|^2$ zu wählen ist."

¹⁰ "Wir wollen diese [...] Funktion im Sinne der von Born in seiner Stoßmechanik [here he cites [8, 9]] vertretenen Auffassung des "Gespensterfeldes" folgendermaßen deuten: Es ist $|\psi(q_1 \dots q_f)|^2 dq_1 \dots dq_f$ die Wahrscheinlichkeit dafür, daß im betreffenden Quantenzustand des Systems diese Koordinaten sich zugleich im betreffenden Volumenelement $dq_1 \dots dq_f$ des Lageraums befinden."

[32, (4a),(5)]. The interpretational part is in Footnote 2 there, which states that (the formula corresponding in modern notation to $\langle A \rangle := \text{Tr } \rho A$) denotes the probability mean ("Wahrscheinlichkeitsmittelwert"). Again there is no reference to measurement.

3.2 Formulations of Born's rule in terms of measurement

As pointed out by WEYL [63, p.2], the derivation from Born's and Jordan's statement does not extend to general operators A , due to noncommutativity and the resulting complementarity. Another consequence of this noncommutativity is that Born's stationary state probability interpretation and Pauli's position probability density interpretation cannot both claim objective status. Therefore, later interpretations relate the notion of value of an observable (being in the n -state, or having position r) more directly to measurement.

The 1927 paper by VON NEUMANN [46], after having noted (on p. 248) the problems resulting from noncommuting quantities that cannot be observed simultaneously, derives axiomatically on p.255 he derives for a theoretical expectation value with natural properties the necessity of the formula $\langle A \rangle := \text{Tr } \rho A$ with Hermitian ρ (his U) of trace 1. This is abstract mathematical reasoning independent of any relation to measurement, and hence belongs to the formal (uninterpreted) core of quantum physics. However, the motivation for his axioms, and hence their interpretation, is taken from a consideration on p.247 of the measurement of values in an ensemble of systems, taking the expectation to be the ensemble mean of the measured values. Specialized to a uniform ("einheitlich") ensemble of systems in the same completely known (pure) state ψ of norm one he then finds on p.258 that $\rho = \psi\psi^*$, giving $\langle A \rangle := \psi^* A \psi$.

In the present terminology, we may phrase von Neumann's interpretation of q-expectation values as follows:

Born's rule (measured expectation form): If a q-observable corresponding to a self-adjoint Hermitian operator A is measured on a system in the pure state ψ (or the mixed state ρ), the results equal on average the q-expectation value $\langle A \rangle := \psi^* A \psi$ (resp. $\langle A \rangle := \text{Tr } \rho A$).

Note that the q-expectation value has a formal meaning independent of the interpretation; the measured expectation form of Born's rule just asserts that measurements result in a random variable whose expectation agrees with the formal q-expectation. To justify the "equal", the average in question cannot be a sample average (where only an approximate equal results, with an accuracy depending on size and independence of the sample) but must be considered as the theoretical expectation value of the random variable.

On a purist note, we can only take finitely many measurements on a system. But the expectation value of a random variable is insensitive to the result of a finite number of realizations. Thus, in the most stringent sense, the expectation form of Born's rule says

nothing at all about measurement. However, the content of the expectation form is roughly the content of the more carefully formulated statement (MI) discussed in Subsection 2.4, specialized to a pure state. (MI) does not have the defect just mentioned.

More conventionally, Born's rule is phrased in terms of measurement results and their probabilities rather than expectations. As part of Born's rule, it is usually stated (see, e.g., [68]) that the results of the measurement of a q-observable exactly equals one of the eigenvalues.

A precise basic form of Born's rule (often augmented by a more controversial collapse statement about the state after a measurement¹¹ not discussed here) is the following, taken almost verbatim from Wikipedia [68].

Born's rule (discrete form): If a q-observable corresponding to a self-adjoint Hermitian operator A with discrete spectrum is measured in a system described by a pure state with normalized wave function ψ then

- (i) the measured result will be one of the eigenvalues λ of A , and
- (ii) the probability of measuring a given eigenvalue λ_i equals $\psi^* P_i \psi$, where P_i is the projection onto the eigenspace of A corresponding to λ_i .

A related statement is claimed to hold for arbitrary spectra with a continuous part, generalizing both the discrete form and the original form.

Born's rule (universal form):¹² If a q-observable corresponding to a self-adjoint Hermitian operator A is measured in a system described by a pure state with normalized wave function ψ then

- (i) the measured result will be one of the eigenvalues λ of A , and
- (ii) for any open interval Λ of real numbers, the probability of measuring $\lambda \in \Lambda$ equals $\psi^* P(\Lambda) \psi$, where $P(\Lambda)$ is the projection onto the invariant subspace of A corresponding by the spectral theorem to the spectrum in Λ .

If the measurement result λ_i is an isolated eigenvalue of A , the universal form reduces to

¹¹For example, in his famous 1930 book, DIRAC [16, p.49] states: *"The state of the system after the observation must be an eigenstate of [the observable] α , since the result of a measurement of α for this state must be a certainty."* In the third edition [17, p.36], he writes: *"Thus after the first measurement has been made, the system is in an eigenstate of the dynamical variable ξ , the eigenvalue it belongs to being equal to the result of the first measurement. This conclusion must still hold if the second measurement is not actually made. In this way we see that a measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured, the eigenvalue this eigenstate belongs to being equal to the result of the measurement."*

A 2007 source is SCHLOSSHAUER [56], who takes the collapse ("jump into an eigenstate") to be part of what he calls the "standard interpretation" of quantum mechanics, but does not count it as part of Born's rule (p.35). On the other hand, LANDAU & LIFSHITZ [33, Section 7] explicitly remark that the state after the measurement is in general not an eigenstate.

¹² The quantum field theory book by WEINBERG [61] pays on p.50 (2.1.7) lip service to the universal form of Born's rule. But the only place where Born's rule is used is on p.135 (3.4.7), where instead the scattering form is employed to get the transition rates for scattering processes. Thus quantum field theory only relies on the scattering form of Born's rule.

the discrete form, since one can take Λ to be an open interval intersecting the spectrum in λ_i only, and in this case, $P(\Lambda) = P_i$.

Using the spectral theorem, it is not difficult to show that the universal form of Born's rule implies the measured expectation form. Conversely, the measured expectation form of Born's rule almost implies the universal form. It fully implies the second part (ii), from which it follows that the first part (i) holds with probability 1 (but not with certainty, as the universal form claims).¹³

Unfortunately, the application of Born's rule to measurement problems in general is highly questionable. Because of the equivalence just mentioned, it is enough to discuss the universal form of Born's rule.

3.3 Limitations of Born's rule

Though usually stated as universally valid, Born's rule has severe limitations. In the universal form, it neither applies to photodetection nor to the measurement of the total energy, just to mention the most conspicuous misfits. Moreover, equating the results of measurements with exact eigenvalues is very questionable when the latter are irrational or (as in the case of angular momentum) multiples of a not exactly known constant of nature. In addition, real measurements rarely produce exact numbers (as Born's rule would require it) but (cf. [48]) numbers that are themselves subject to uncertainty. Because of these limitations and the inherent ambiguities in specifying what constitutes a measurement¹⁴ and what qualifies as a measurement result, subsequent derivations can never claim universal validity either.

Problems with Born's rule include:

1. At energies below the dissociation threshold (i.e., where the spectrum of the Hamiltonian H , the associated q-observable, is discrete), energy measurements of a system almost never yield an exact eigenvalue of H . For example, nobody knows the exact value of the Lamb shift, a difference of eigenvalues of the Hamiltonian of the hydrogen atom; the (reasonably) precise measurement was even worth a Nobel prize (1955 for

¹³ This is not just hair splitting. The difference between probability 1 and certainty can be seen by noting that the probability that a random number drawn uniformly from $[0, 1]$ is irrational with probability 1, while measurements usually produce rational numbers.

In general, the expectation form certainly allows finitely many exceptions to (i), and hence says, strictly speaking, nothing at all about the possible measurement results. (This is well illustrated in the story "The Metaphysician's Nightmare" by the logician Bertrand RUSSELL [53]: "There is a special department of Hell for students of probability. In this department there are many typewriters and many monkeys. Every time that a monkey walks on a typewriter, it types by chance one of Shakespeare's sonnets.") However, the application to spin measurements and to quantum information theory requires that (i) holds at least for binary spectra. Hence the expectation form is slightly deficient.

¹⁴This is a highly nontrivial problem in quantum statistical mechanics; cf. ALLAHVERDYAN et al. [1].

Willis Lamb). Indeed, the energy levels of most realistic quantum systems are only inaccurately known.

2. In particular, Born's rule does not apply to the total energy of a composite system, one of the key q-observables¹⁵ in quantum physics, since the spectrum is usually very narrowly spaced and precise energy levels are known only for the simplest systems in the simplest approximations. Therefore Born's rule cannot be used to justify the canonical ensemble formalism of statistical mechanics; it can at best motivate it.
3. The same holds for the measurement of masses of relativistic particles with 4-momentum p , which never yield exact eigenvalues of the mass operator $M := \sqrt{p^2}$. Indeed, the masses of most particles are only inaccurately known.
4. When a particle has been prepared in an ion trap (and hence is there with certainty), Born's rule implies a tiny but positive probability that at an arbitrarily short time afterwards it is detected a light year away.¹⁶ In a similar spirit, HEISENBERG [24, p.25] wrote in 1930:¹⁷ *"This result is stranger than it seems at first glance. As is well known, $\psi^*\psi$ diminishes exponentially with increasing distance from the nucleus; there is thus always a small but finite probability of finding the electron at a great distance from the center of the atom."* Thus $|\psi(x)|^2$ cannot be the exact probability density for being detected at x .
5. This argument against the exact probability density interpretation of $|\psi(x)|^2$ works even for relativistic particles in the multiparticle framework of KEISTER & POLYZOU [29].
6. A no-go theorem for exact measurement by WIGNER [67] rules out projective measurements of a particle being in a given region, since the corresponding projector does not commute with all additive conserved quantities. See also OZAWA [49] and ARAKI & YANASE [3].
7. Many measurements in quantum optics are POVM measurements [69], i.e., described by a positive operator-valued measure. These follow a different law of which Born's rule is just a very special case where the POVM is actually projection-valued. (In general, the POVM law can be derived from Born's rule applied to a fictitious von Neumann experiment in an extended Hilbert space. This shows its consistency with

¹⁵ "we shall assume the energy of any dynamical system to be always an observable" (DIRAC [16, p.38])

¹⁶ Indeed, for a single massive particle, Born's rule states that $|\psi(x,t)|^2$ is the probability density for locating at a given time t the particle at a particular position x anywhere in the universe, and the Fourier transform $|\tilde{\psi}(p,t)|^2$ is the probability density for locating at a given time t the particle with a particular momentum p . In the present case, the position density has bounded support, so by a basic theorem of harmonic analysis, the momentum density must have unbounded support. This implies the claim.

¹⁷German original: *"Das Resultat ist aber merkwürdiger, als es im ersten Augenblick den Anschein hat. Bekanntlich nimmt $\psi^*\psi$ exponentiell mit wachsendem Abstand vom Atomkern ab. Also besteht immer noch eine endliche Wahrscheinlichkeit dafür, das Elektron in sehr weitem Abstand vom Atomkern zu finden."*

Born's rule but still disproves the latter for the actual physical states in the smaller physical Hilbert space.)

8. Born's rule does not cover the multitude of situations where typically only single measurements of a q-observable are made. In particular, Born's rule does not apply to typical macroscopic measurements, whose essentially deterministic predictions are derived from statistical mechanics.
9. The measurement of quantum fields is not covered by Born's rule. These are q-observables depending on a space-time argument, and one can prepare or measure events at any particular space-time position at most once. Thus it is impossible to repeat measurements, and the standard statistical interpretation in terms of sufficiently many identically prepared systems is impossible.
10. Many things physicists measure have no simple interpretation in terms of a Born measurement. Examples include spectral lines and widths, particle masses and life times, chemical reaction rates, or scattering cross sections. Often lots of approximate computations are involved.

To uphold Born's rule in view of points 1.-3., one would have to consider the concept of measurement that of a fictitious, infinitely precise measurement.¹⁸

Points 4 and 5 above imply that the wave function and hence the density operator encode in their basic operational interpretation highly nonlocal information. Thus nonlocality is explicitly built into the very foundations of quantum mechanics *as conventionally presented*. Processing nonlocal information, it is no surprise that standard quantum mechanics defined by the Schrödinger equation violates the conclusions of Bell type theorems (cf. the discussion and references in NEUMAIER [37]). It already violates their assumptions!

Points 4 and 5 also show that at finite times (i.e., outside its use to interpret asymptotic S-matrix elements), Born's rule cannot be strictly true in relativistic quantum field theory, and hence not in nature.

3.4 The domain of validity of Born's rule

As we have seen, Born's rule has, like any other statement in physics, its domain of validity but leads to problems when applied outside this domain.¹⁹ From an analysis of many different q-observables and measurement protocols, it seems that the discrete form of Born's

¹⁸In particular, this would exclude a subjective interpretation of measurement results and associated probabilities in terms of the experimenter's knowledge.

¹⁹Progressing from the Born rule to so-called positive operator valued measures (POVMs) is already a big improvement, commonly used in quantum optics and quantum information theory. These are adequate for measurements in the form of clicks, flashes or events (particle tracks) in scattering experiments, and perhaps only then.

But these still do not cover measurements of energy, or of the Lamb shift, or of particle form factors.

rule needs four conditions for its validity. It is valid precisely for measuring q-observables

1. with only discrete spectrum,
2. measured over and over again in identical states (to make sense of the probabilities), where
3. the difference of adjacent eigenvalues is significantly larger than the measurement resolution, and where
4. the measured value is adjusted to exactly match the spectrum, which must be known exactly prior to the measurement.

The universal version has similar limitations also when restricted to purely continuous spectra; in this case it seems to be valid only in Born's original scattering form.

3.5 What is a state?

In physics, the state of a physical system (whether classical or quantum) gives a complete description of the system at a given time. The following is a concise formulation of this:

(S1) The state of a system (at a given time) encodes everything that can be said (or "can be known") about the system at this time, including the possible predictions at later times, and nothing else.

Thus every property of the system can (in principle) be computed from its state.

For a complex system, knowledge about the whole system is usually obtained by collecting knowledge about its various parts. This makes sense only if we require in addition,

(S2) Every property of a subsystem is also a property of the whole system.

Indeed, not knowing something about the subsystem means not knowing everything about the system as a whole, and hence not knowing the precise state of the system.

Thus common sense dictates that a sound, observer-independent interpretation of quantum physics should satisfy (S1) and (S2).

Now (S2) says that the state of the full system determines all properties of any of its subsystems. hence it determines by (S1) the state of each subsystem to the last detail. Thus we conclude:

(S3) The state of a system determines the state of all its subsystems.

A macroscopic body should therefore have a valid microscopic quantum description – a quantum state – that determines all observable properties on every level. In particular, an approximate hydromechanical classical description for the most important observable properties, namely the q-expectation values of the fields, must be obtainable from this exact quantum state. (The process to achieve this is usually called coarse graining.)

Property (S3) must hold for logical reasons even though in practice we may never know the precise state of the system and/or the subsystems. Indeed, we usually know only very little information about any system, unless the latter is so tiny that it can be fully described by very few parameters.

Unfortunately, none of the mainstream versions of the interpretation of quantum mechanics (i.e., those not invoking hidden variables) are anywhere presented in a form that would satisfy our conclusion (S3). Since the deficiency always has the same root – the treatment of the density operator as representing a state of incomplete knowledge, a statistical mixture of pure states – it is enough to discuss one specific interpretation. We shall look at the interpretation given in the very influential treatise of theoretical physics by LANDAU & LIFSHITZ [33, 34]. They start their discussion of quantum mechanics with a particular version of Born’s rule:

[33, p.6] *”The basis of the mathematical formalism of quantum mechanics lies in the proposition that the state of a system can be described by a definite (in general complex) function $\Psi(q)$ of the coordinates. The square of the modulus of this function determines the probability distribution of the values of the coordinates: $|\Psi|^2 dq$ is the probability that a measurement performed on the system will find the values of the coordinates to be in the element dq of configuration space. The function Ψ is called the wave function of the system. [...] If the wave function is known at some initial instant, then, from the very meaning of the concept of complete description of a state, it is in principle determined at every succeeding instant.”*

Thus in terms of our formal core, the complete description of the system is declared by Landau & Lifshitz to be a pure state, and the properties of the system are declared to be the probabilities of potential measurement results. They then consider parts (subsystems) and observe:

[33, p.7] *”Let us consider a system composed of two parts, and suppose that the state of this system is given in such a way that each of its parts is completely described.†”*

Footnote: *”† This, of course, means that the state of the whole system is completely described also. However, we emphasize that the converse statement is by no means true: a complete description of the state of the whole system does not in general completely determine the states of its individual parts”*

Thus they explicitly deny (S3). In fact, except in the special case discussed in the context of the above quote – where the state factors into a tensor product of states of the subsystems – *they do not indicate at all how the state of a system and that of its parts are related.* More mysteriously, nowhere in the literature seems to be a discussion that would tell us anything on the formal level about the relationship between the pure state of a system and the pure state of a subsystem. There seems to be no such relation, except in the idealized, separable case mentioned above, usually assumed to be valid only before an interaction happens.

But this would mean that the quantum state of a physics lab has nothing to do with the quantum states of the equipment in it, and of the particles probed there! This is *very* strange for a science such as physics that studies large systems primarily by decomposing them into its simple constituents and infers properties of the former from collective properties of the latter.

This truly unacceptable situation shows that there is something deeply wrong with the traditional interpretations.²⁰ It is no surprise that this leads to counterintuitive paradoxes in situations – such as experiments with entangled photons – where a larger (e.g., 2-photon) system is prepared but its constituents (here 2 single photons) are observed.

Of course, we can never know the exact quantum state of a physics lab or a piece of equipment. Because of that it has become respectable to interpret quantum mechanics not in terms of what is but in terms of what is known to the person modeling a physical system. The system state then becomes a complete description no longer of the physical system but of the knowledge available.²¹ Uncertainty about the pure state is then modeled as a probability distribution for being in a pure state. Averaging with corresponding weight leads to more general mixed states described by density operators. In this context, LANDAU & LIFSHITZ write:

[34, p.16] *”The quantum-mechanical description based on an incomplete set of data concerning the system is effected by means of what is called a density matrix [...] The incompleteness of the description lies in the fact that the results of various kinds of measurement which can be predicted with a certain probability from a knowledge of the density matrix might be predictable with greater or even complete certainty from a complete set of data for the system, from which its wave function could be derived.”*

Based on this, they derive the interpretation of the q-expectation $\langle A \rangle := \text{Tr } \rho A$ as the expectation value of A in a mixed state ρ :

[34, p.17] *”The change from the complete to the incomplete quantum-mechanical description of the subsystem may be regarded as a kind of averaging over its various ψ states. [...] the mean value \bar{f} becomes the trace (sum of diagonal elements) of this operator”*

However, on the next page they call their description an illustration only, denying it any trace of reality:

[34, p.18] *”It must be emphasised that the averaging over various ψ states, which we have used in order to illustrate the transition from a complete to an incomplete quantum-mechanical description, has only a very formal significance. In particular, it would be quite*

²⁰In addition, there are the traditional difficulties of interpretations of quantum mechanics, well summarized in Section 3.7 of the quantum mechanics textbook by WEINBERG [62].

²¹Whose knowledge this is is usually not addressed; presumably it is the knowledge of the person creating the mathematical model of the quantum system. Taken at face value, this would make the system state a function of the mental state of the modeler’s mind – another truly unacceptable situation.

incorrect to suppose that the description by means of the density matrix signifies that the subsystem can be in various ψ states with various probabilities and that the averaging is over these probabilities. Such a treatment would be in conflict with the basic principles of quantum mechanics.

The states of a quantum-mechanical system that are described by wave functions are sometimes called pure states, as distinct from mixed states, which are described by a density matrix. Care should, however, be taken not to misunderstand the latter term in the way indicated above.

The averaging by means of the statistical matrix according to (5.4) has a twofold nature. It comprises both the averaging due to the probabilistic nature of the quantum description (even when as complete as possible) and the statistical averaging necessitated by the incompleteness of our information concerning the object considered. For a pure state only the first averaging remains, but in statistical cases both types of averaging are always present. It must be borne in mind, however, that these constituents cannot be separated; the whole averaging procedure is carried out as a single operation, and cannot be represented as the result of successive averagings, one purely quantum-mechanical and the other purely statistical.”

Thus Landau and Lifschitz reject the subjective, knowledge-based view, as one cannot divide the information contained in the density operator into an objective, pure part corresponding to the objective properties of the system and a statistical part accounting for lack of knowledge.

But this also means that their derivation of the interpretation of the q-expectation $\langle A \rangle := \text{Tr} \rho A$ as expectation value is invalid, being based on an invalid illustration only. Note that this formula is heavily used in quantum statistical mechanics and quantum field theory. It is often applied there in contexts where no measurement at all is involved and when it is not even clear how one should measure the operators in question.²² Indeed, most of quantum statistical mechanics is not concerned with measurement at all. In all these cases the connection to measurement and hence to Born’s rule is absent, and even the hand-waving ”illustrative” derivation given is spurious, as the items going into the derivation are never actually measured.

On the other hand, the use of the density operator is central to quantum statistical mechanics. The fact that the latter predicts qualitatively and quantitatively the thermodynamics of macroscopic systems shows that **the density operator contains objective, knowledge-independent information, and is the true carrier of the state information in quantum physics.** This is one of the reasons why the description of the formal core of quantum physics presented in Subsection 2.1 featured the density operator as basic. Pure states then appear as idealizing approximation under the conditions discussed in Subsection 2.2.

It is very remarkable that thermodynamics provides an alternative interpretation of the

²²We mentioned already in Subsection 3.3 the problems with interpreting measurements of the energy, corresponding to the operator H figuring in all of quantum statistical mechanics.

q-expectation $\langle A \rangle := \text{Tr } \rho A$ – not as an expectation value, but as the macroscopic, approximately measured value of A . This is the germ of the thermal interpretation of quantum physics to be discussed in Part II [40].

3.6 Pure states in quantum field theory

That pure states cannot have a fundamental meaning can also be seen from the perspective of quantum field theory. It is a very little known fact that, in any interacting relativistic quantum field theory, the notion of a pure state loses its meaning. Results from algebraic quantum field theory (cf. YNGVASON [71, p.12]) imply that all local algebras induced by a relativistic quantum field theory on a causal diamond (an intersection of a future cone and a past cone with nonempty interior) are factors of type III_1 in von Neumann’s classification of factors as refined by CONNES [14]. Picking such a causal diamond containing our present planetary system implies that we may assume the algebra of observables currently accessible to mankind to be such a factor of type III_1 . Remarkably, such a C^* -algebra \mathbb{A} has no pure states [71, p.14].²³ Therefore, in these representations, one cannot rigorously argue about states by considering partial traces in nonexistent pure states! This shows that pure states must be the result of a major approximating simplification, and not something fundamental.

Note that \mathbb{A} has infinitely many unitarily inequivalent irreducible representations on Hilbert spaces (corresponding to the different superselection sectors of the theory). But in each such representation, the algebra \mathbb{A} of bounded q-observables is vanishingly small compared to the algebra of all bounded operators. A vector state in the Hilbert space \mathbb{H} of an irreducible representation of a local algebra \mathbb{A} of type III_1 (which is a pure state in \mathbb{H}) can therefore still be mixed as a state of \mathbb{A} .²⁴ The vector state is guaranteed to be pure only relative to the algebra of all bounded operators on \mathbb{H} . But this algebra is far bigger than the algebra \mathbb{A} , and contains lots of operators that have no interpretation as q-observables. This is the essential difference to the case of type I algebras, realizable in a Fock space, which have many pure states. These algebras are the local algebras of free quantum field theories, and only encode systems of noninteracting particles.

Thus what breaks down in quantum field theory is the simple equation ”q-observable = self-adjoint Hermitian linear operator”. Once this equation is broken, the question whether a state is pure becomes dependent on the precise specification of which operators are q-observables. In gauge theories the situation is further complicated by the fact that the local

²³An explicit example of a factor of type III_1 involving an infinite array of spin-1/2 particles is given in equations (27) and (29) of YNGVASON [71].

²⁴For example, for an arbitrary mixed state of a Hilbert space \mathbb{H}_0 , the GNS construction produces another Hilbert space in which this state is pure. Note that in quantum physics, the GNS construction is of limited value only, as this Hilbert space depends on the state one started with, while standard quantum mechanics works with pure states contained in a **fixed** Hilbert space. One therefore needs a distinguished state to define a Hilbert space. Now the only distinguished state in quantum field theory is the vacuum state. But for gauge theories such as quantum electrodynamics, the Hilbert space corresponding to this vacuum representation does not contain any charged state!

algebras have a nontrivial center consisting of charges that in each irreducible representation are represented trivially. Thus a single irreducible representation on a single Hilbert space (corresponding to a single superselection sector) is no longer sufficient to characterize the complete algebra of local q -observables.

To give up the assumption that every bounded self-adjoint operator is a q -observable has serious consequences for the interpretation of quantum physics. Indeed, a test for a pure state is in terms of q -observables an observation of the orthogonal projector to the subspace spanned by the state. If this is not a q -observable then it is in principle impossible to make this test. Thus testing for being in a pure state is impossible, since these are no longer physical propositions.

So one cannot decide whether a system is in a pure state. To assume this is thus a metaphysical act, and one can dispense with it without any loss of information. But then the conventional form of Born's rule hangs in the air, and all traditional interpretations break down completely, since they start with Born's rule for pure states and derive everything else from it.

4 Requirements for good foundations

The ordinary language, (spiced with technical jargon for the sake of conciseness) is thus inseparably united, in a good theory, with whatever mathematical apparatus is necessary to deal with the quantitative aspects. It is only too true that, isolated from their physical context, the mathematical equations are meaningless: but if the theory is any good, the physical meaning which can be attached to them is unique.

Leon Rosenfeld, 1957 [52, p.41]

I feel induced to contradict emphatically an opinion that Professor L. Rosenfeld has recently uttered in a meeting at Bristol, to the effect that a mathematically fully developed, good and self-consistent physical theory carries its interpretation in itself, there can be no question of changing the latter, of shuffling about the concepts and formulae.

Erwin Schrödinger, 1958 [58, p.170]

A great physical theory is not mature until it has been put in a precise mathematical form, and it is often only in such a mature form that it admits clear answers to conceptual problems.

Arthur Wightman, 1976 [66, p.158]

4.1 Foundations independent of measurement

to me it must seem a mistake to permit theoretical description to be directly dependent upon acts of empirical assertions

Albert Einstein, 1949 [19]

Measurement should not figure in the foundations of physics; the case for this was vividly made by BELL [7]. The analysis in Section 3 indeed shows that actual measurement practice is in conflict with the traditional foundations, due to a far too idealized view of measurement.

Thus we are led to inquire how foundations independent of measurement could look like. This can be studied by looking at the modern account of the oldest of the physical sciences, Euclidean geometry.

For Euclidean geometry, considered as a branch of physics, there is a complete consensus about how theory and reality correspond (on laboratory scales).

One first defines the corresponding calculus and names the quantities that can be calculated from quantum mechanical models (or models of the theory considered) with the appropriate names from experimental geometry. Thus, initially, a circle was a material object with a round shape, and the mathematical circle was an abstraction of these.

The Pythagoreans (and later Descartes and Hilbert with even more precision) then developed a theory that gives a precise formal meaning to all the geometrical concepts. This is pure mathematics, today encoded in textbook linear algebra and analytic geometry. The theory and the nomenclature were *developed* with the goal of enabling this identification in a way consistent with tradition.

Starting with Plato, the theory took precedence, defining the perfect concept. What was found in reality was viewed as an approximate, imperfect realization of the theoretical concept.

This was done by declaring anything in real life resembling an ideal point, line, plane, circle, etc., to be a point, line, plane, circle, etc., if and only if it can be assigned in an approximate way (determined by the heuristics of traditional measurement protocols, whatever they are) the properties that the ideal point, line, plane, circle, etc., has, consistent to the assumed accuracy with the deductions from the theory. If the match is not good enough, we can explore whether an improvement can be obtained by modifying measurement protocols (devising more accurate instruments or more elaborate error-reducing calculation schemes, etc.) or by modifying the theory (to a non-Euclidean geometry, say, which uses the same concepts but assumes slightly different properties relating them).

No significant philosophical problems are left; lucid, intuitive, and logically impeccable foundations for Euclidean geometry have been established in this way. This indicates the maturity of Euclidean geometry as a scientific discipline.

Thus once the theory is mature, the identification with real life is done in terms of the formal, purely mathematical theory developed, giving an interpretation to the theory. In this way, physics inherits the clarity of mathematics, the art and science of precise concepts and relations.

In particular, with the Lagrangian and Hamiltonian formulations, classical mechanics has

also reached the status of maturity, and hence is perceived by most physicists as clear and philosophically unproblematic.

To define quantum physics (or any other physical theory) properly, including a logically impeccable interpretation, one should therefore proceed as in Euclidean geometry and classical mechanics.

One first needs to define the corresponding calculus; this has already been done in Section 2.1. Then one has to name the quantities that can be calculated from quantum mechanical models (or models of the theory considered) with the appropriate names the experimental physicists use for organizing their data.

One can then develop a theory that gives a precise formal meaning to the concepts physicists talk about. This is pure mathematics – the shut-up-and-calculate part of quantum physics. Finally, an **interpretation** to the theory, i.e., the identification with real life, is given *in terms of the formal theory developed* by means of

(CC) Callen’s criterion: Operationally, a system is in a given state if its properties are consistently described by the theory for this state.

This generalizes the way H.B. Callen justified phenomenological equilibrium thermodynamics in his famous textbook (CALLEN [12]), where he writes on p.15: *”Operationally, a system is in an equilibrium state if its properties are consistently described by thermodynamic theory.”* At first sight, this sounds like a circular definition (and indeed Callen classifies it as such). But a closer look shows there is no circularity since the formal meaning of ‘consistently described by thermodynamic theory’ is already known. The operational definition simply moves this formal meaning from the domain of theory to the domain of reality by defining when a real system deserves the designation ‘is in an equilibrium state’. In particular, this definition allows one to determine experimentally whether or not a system is in equilibrium.

For quantum physics, Callen’s criterion implies that we declare anything in real life resembling an ideal photon, electron, atom, molecule, crystal, ideal gas, etc., to be a photon, electron, atom, molecule, crystal, ideal gas, etc., if and only if it can be assigned in an approximate way (determined by the heuristics of traditional measurement protocols, whatever that is) the properties that the ideal photon, electron, atom, molecule, crystal, ideal gas, etc., has, consistent to the assumed accuracy with the deductions from the theory.

This identification process is fairly independent of the way measurements are done, as long as they are capable to produce the required accuracy for the matching, hence carries no serious philosophical difficulties.

Of course, any successful theory must be crafted in such a way that it actually applies to reality – otherwise the observed properties cannot match the theoretical description. On

the other hand, as the quote by Callen emphasizes, **we already need the theory to define precisely what it is that we observe.**

As a result, theoretical concepts and experimental techniques complement each other in a way that, if a theory is reaching maturity, it has developed its concepts to the point where they are a good match to reality. We then say that:

(R) Something in real life 'is' an instance of the theoretical concept if it matches the theoretical description sufficiently well.

It is not difficult to check that this holds not only in physics but everywhere where we have clear concepts about some aspect of reality.

If the match between theory and observation is not good enough, we can explore whether an improvement can be obtained by modifying measurement protocols (devising more accurate instruments or more elaborate error-reducing calculation schemes, etc.) or by modifying the theory (to a hyper quantum physics, say, which uses the same concepts but assumes slightly different properties relating them).

Having established informally that the theory is an appropriate model for the physical aspects of reality, one can study the measurement problem rigorously on this basis: One declares that a real **detector** (in the sense of a complete experimental arrangement including the numerical postprocessing of raw results that gives the final result) performs a real **measurement** of an ideal quantity if and only if the following holds: Modeling the real detector as a macroscopic quantum system (with the properties assigned to it by statistical mechanics/thermodynamics) predicts raw measurements such that, in the model, the numerical postprocessing of raw results that gives the final result is in sufficient agreement with the value of the ideal quantity in the model.

Then measurement analysis is a scientific activity like any other rather than a philosophical prerequisite for setting up a consistently interpreted quantum physics. Indeed, this is the way high precision experiments are designed and analyzed in practice.

4.2 What is a measurement?

Good foundations including a good measurement theory should be able to properly justify this informal consensus by defining additional formal concepts about what constitutes measurement. To be satisfying, these must behave within the theory just as their informal relatives with the same name behave in reality. Then instrument builders may use the theory to inform themselves of what can possibly work, and instrument calibration assumes the laws of physics to hold.

Thus measurement must be grounded in theory, not – as in the traditional foundations – the other way round! In complete foundations, there would be formal objects in the

mathematical theory corresponding to all informal objects discussed by physicists, including those used when designing and performing measurements. Only then talking about the formal objects and talking about the real objects is essentially isomorphic. We are currently far from such complete foundations.

To understand the precise meaning of the notion of measurement we look at measurement in the context of classical physics and chemistry. There are two basic kinds of measurements, destructive measurements and nondestructive measurements.

Nondestructive measurements either leave the state of the object measured unchanged (such as in the measurement of the length of a macroscopic object) or modify it temporarily during the measurement (e.g., temporarily deforming it to measure the stiffness) in such a way that the object returns to its original state after the measurement is completed. **Destructive measurements** permanently change the state of the object measured, usually by destroying all or part of it during the measurement process. Examples are the determination of the age of an archeological artifact by dendrochronology, or many traditional methods of finding the chemical composition of a material.

In both cases, the measurement gives some **posterior** information about the **prior** state, i.e., the state of the object before the start of the measurement process. In case of destructive measurements, it also gives some information about the products of the destruction, from which properties of the prior states are deduced by reasoning.

A characteristic context of destructive measurements is the presence of a large, sufficiently homogeneous object. Tiny parts of it are subjected to destructive measurements to discover their relevant properties. The homogeneity of the object then implies that the properties deduced from the destructive measurements are also properties of the remainder of the object. Thus destructive measurements of a tiny fraction of a homogeneous object give information about the whole object, including its unmeasured part. By our definition, we obtain in this way a nearly nondestructive measurement of the whole object.

Alternatively, a large number of essentially identical objects are present, a few of which are subjected to a destructive measurement. The results of the measurement are then taken as being representative of the properties of the unmeasured objects. In case the measurements on the objects measured do not agree, one can still make statistical statements about the unmeasured objects, approximately valid within the realm of validity of the law of large numbers. However, this no longer gives valid information about a single unmeasured object, but only information about the whole population of unmeasured objects. Thus one may regard the measurement on multiple trial objects as a measurement of the state of the whole population.

Based on this analysis we conclude:

(M) A property P of a system S (encoded in its state) has been measured by another system, the **detector** D , if at the time of completion of the measurement and a short time

thereafter (long enough that the information can be read by an observer) the detector state carries enough information about the state of the measured system S at the time when the measurement process begins to deduce with sufficient reliability the validity of property P at that time.

4.3 Beables

*The scientist [...] appears as **realist** insofar as he seeks to describe a world independent of the acts of perception; as **idealist** insofar as he looks upon the concepts and theories as the free inventions of the human spirit (not logically derivable from what is empirically given); as **positivist** insofar as he considers his concepts and theories justified only to the extent to which they furnish a logical representation of relations among sensory experiences. He may even appear as **Platonist** or **Pythagorean** insofar as he considers the viewpoint of logical simplicity as an indispensable and effective tool of his research. [original **bold** preserved]*

Albert Einstein, 1949 [19]

One of the basic problems with the traditional interpretations of quantum mechanics is the difficulty to specify precisely what counts as real. The physics before 1926 was explicitly about discovering and objectively describing the true, reliably repeatable features of nature, seen as objectively real.

After the establishment of modern quantum physics, the goal of physics can (according to the traditional interpretations of quantum mechanics) only be much more modest, to systematically describe what physicists measure. Nonetheless, physics continues to make objective claims about reality that existed long before a physicist performed the first measurement, such as the early history of the universe, the composition of distant stars and galaxies of which we can measure not more than tiny specks of light, the age of ancient artifacts dated by the radio carbon method. Physics also makes definite statements about the distant future of our solar systems – independent of anyone being then around to measure it.

Thus there is a fundamental discrepancy between what one part of physics claims and what the traditional interpretations of quantum mechanics allows one to claim. This discrepancy was discussed in a paper by BELL [6], where he writes that quantum mechanics “*is fundamentally about the results of ‘measurements’, and therefore presupposes in addition to the ‘system’ (or object) a ‘measurer’ (or subject). [...] the theory is only approximately unambiguous, only approximately self-consistent. [...] it is interesting to speculate on the possibility that a future theory will not be intrinsically ambiguous and approximate. Such a theory could not be fundamentally about ‘measurements’, for that would again imply incompleteness of the system and unanalyzed interventions from outside. Rather it should again become possible to say of a system not that such and such may be observed to be so but that such and such be so. The theory would not be about ‘observables’ but about ‘beables’. These beables [...] should, on the macroscopic level, yield an image of the everyday classical world, [...] the familiar language of everyday affairs, including laboratory procedures, in*

which objective properties – beables – are assigned to objects.”

To find beables we note that the problems created by quantum mechanics are absent in classical mechanics. Thus it seems that classical objects **exist** in the sense that they have objective properties that qualify as beables. The classical regime is usually identified with macroscopic physics, where length and time scales are long enough that the classical approximation of quantum mechanics is accurate enough to be useful. This suggests that we look at the visible parts of quantum experiments.

Most experiments done to probe the foundations of quantum physics are done using optical devices. In quantum optics experiments, both sources and beams are extended macroscopic objects describable by quantum field theory and statistical mechanics. For example, a laser beam is simply a coherent state of the quantized electromagnetic field, concentrated in a neighborhood of a line segment in space.

The sources have properties independent of measurement, and the beams have properties independent of measurement. These are objects described by quantum field theory. For example, the output of a laser (before or after parametric down conversion or any other optical processing) is a laser beam, or an arrangement of highly correlated beams. These are in a well-defined state that can be probed by experiment. If this is done, they are always found to have the properties ascribed to them by the preparation procedure. One just needs sufficient time to collect the information needed for a quantum state tomography. The complete state is measurable in this way, reproducibly, to any given accuracy.

Neither the state of the laser nor of the beam is changed by one or more measurements at the end of the beam. Moreover, these states can be found to any desired accuracy by making sufficiently long and varied measurements of the beam; how this is done is discussed in quantum optics under the name of quantum tomography.

Thus these properties exist independent of any measurement – just as the moon exists even when nobody is looking at it. They can be found through diligent measurement, just as properties of distant stars and galaxies. They behave in every qualitative respect just like classical properties of classical objects.

On the other hand, measuring individual particles is an erratic affair, and unless experiments are specially tuned (“non-demolition measurements”) they change the state of the individual particles in an unpredictable way, so that – like in a classical destructive measurement – their precise state before measurement can never be ascertained. Only probabilities for their collective behavior can be given by averaging over many observations of different realizations.

For example, the analysis of experimental particle collisions is based on measuring the momentum and charge of many individual collision products. But *individual* collision events (the momentum and charge of the individual collision products) are not predicted by the theory – only the possibilities and their collective statistics, their distribution in a *collection*

of equally prepared events. Indeed, probabilities mean nothing for a single collision. What does it mean that the particular collision event recorded at a particular time in a particular place is obtained with probability 0.07? Nothing at all; the single simply happened and has no associated probability. A statement about probabilities is always a statement about a process that can be repeated many times under essentially identical conditions.

Thus we have found a class of beables: the densities, intensities, and correlation functions used to describe optical fields. And we have found a class of non-beables: the individual particles. The beables are computable from quantum statistical mechanics and quantum field theory as expectations – not as eigenvalues of q -observables! They are associated with quantum statistical mechanics and quantum field theory on the level of fields – not on the level of individual particles.

Therefore, sources and beams are much more real than particles. The former, not the latter, must be the real players in solid foundations.

The goal of fundamental physics to understand things at the smallest scales possible resulted in quantum field theory, where fields, not particles, play the fundamental role. Particles appear only as asymptotic excitations of the fields. Quantum field theory (or perhaps an underlying theory such as string theory of which quantum field theory is an approximation) is supposed to determine the behavior at all scales, and hence lead to insights into the world at large.

Indeed, as we have seen, the basis of our perception of an objective reality is statistical mechanics and field theory – not few-particle quantum mechanics! The inappropriate focus on the particle aspect of quantum mechanics created the appearance of mystery; common sense is restored by focussing instead on the field aspect.

The deeper reason for this is that from a fundamental point of view, the particle concept is a derived, approximate concept derived from the more basic concept of an interacting relativistic quantum field theory. The quantum particle concept makes mathematical sense only under very special circumstances, namely in those where a system actually behaves like particles do – when they can be considered as being essentially free, as before and after a scattering event in a dilute gas or a particle collider ring. The particle concept makes intuitive sense only under the same circumstances. (This is discussed in more detail in Part III [41] of this series of papers.)

We conclude from our discussion that

(F) Fields are real and have associated beables, given by expectation values.

Conclusion (F) is one of the basic assumptions of the thermal interpretation of quantum physics to be discussed in Part II [40] of this series of papers. It is routinely used in equilibrium and nonequilibrium statistical thermodynamics (see, e.g., CALZETTA & HU [13]).

However, quite early in the history of modern quantum physics, EHRENFEST [18] found a clean and exact relation between the dynamical laws of classical mechanics and quantum mechanics. The **Ehrenfest equation**²⁵ states that

$$\frac{d}{dt}\langle A \rangle_t = \langle H \angle A \rangle_t, \quad (4)$$

where $\langle \cdot \rangle_t$ is the expectation in the state at time t and

$$H \angle A := \frac{i}{\hbar}[H, A]. \quad (5)$$

The implication for the interpretation of quantum physics (to be discussed in Part II [40]) seem to have gone unnoticed in the literature, probably because of the very strong tradition that placed an unreasonable notion of quantum measurement at the very basis of quantum physics.

4.4 What is a particle?

The preceding featured beams of light, conventionally associated with massless particles, the photons, and showed that the beams themselves are far more real than the photons that they are supposed to contain.

An independent indication of the unreal status of photons is provided by an analysis of the photoelectric effect, that faint coherent laser light falling on a photosensitive plate causes randomly placed detection events following a Poisson distribution. Conventionally, this effect is ascribed to the particle nature of light, and each detection event is taken as a proof that a photon arrived. Upon closer analysis, however, the detection events are found to be artifacts caused by the quantum nature of the photosensitive plate. This must be the case because the analysis can be done in a model of the process in which no photons exist. Such an analysis is done, e.g., in Sections 9.1-9.5 of MANDEL & WOLF [35], a standard reference for quantum optics. It is a proof that detection events happen in the detector without photons being present. Hence one cannot tell from a detection event whether the cause was a photon or a classical field. But if the detectors cannot even distinguish an external classical field from an impinging photon in the theoretical analysis, based on which analysis should an experimenter decide?

In the model, the electron field of the detector responds to a classical external electromagnetic radiation field by emitting electrons according to Poisson-law probabilities. Thus the quantum detector produces discrete Poisson-distributed clicks, although the source is completely continuous. The state space of this quantum system consists of multi-electron

²⁵It is a pity that Ehrenfest did not develop this equation to the point where it would have amounted to an interpretation of quantum physics. This could have avoided a lot of the subsequent confusion.

states only. Thus the multi-electron system (followed by a macroscopic decoherence process that leads to the multiple localization of the emitted electron field) is responsible for the creation of the discrete detection pattern.²⁶

Quantum electrodynamics is of course needed to explain special quantum effects of light revealed in modern experiments, but not for the photoelectric effect. Indeed, finer analysis reveals that beams in nonclassical states may give a counting statistics significantly different from that of the classical analysis. But this only shows that the beam description needs quantum field theory (where people conventionally use the language of photons), not that there must be actual particles called photons.

It may seem, however, that the reality of individual *massive* particles is established beyond doubt through the observation of particle tracks in bubble chambers and other path-tracking devices. But are the observed "tracks" guaranteed to be traces of particles?

The paper by SCHIRBER [54] discusses essentially the same phenomenon in a fully classical context, where a bullet is fired into a sheet of glass and produces a large number of radial cracks in random directions. This is shown in the first figure there, whose caption says, "*The number of cracks produced by a projectile hitting a glass sheet provides information on the impactor's speed and the properties of the sheet.*" In the main text, we read "*A projectile traveling at 22.2 meters per second generates four cracks in a 1-millimeter-thick sheet of Plexiglas. [...] A 56.7 meter-per-second projectile generates eight radial cracks in the same thickness Plexiglass sheet as above.*" (See also FALCAO & PARISIO [20] and VANDENBERGHE & VILLERMAUX [60].)

We see that the discrete, random detection events (the cracks) are a manifestation of broken symmetry when something impacts a material that – unlike water – cannot respond in a radially symmetric way. Randomness is inevitable in the breaking of a radial symmetry into discrete events. The projectile creates an outgoing spherical stress wave in the plexiglas and produces straight cracks. In fact, once initiated, the growth of a crack in a solid is not very different from the growth of a track in a bubble chamber, except that the energies and time scales are quite different. Only the initiation is random.

Would observed tracks in a high energy collision energy experiment prove without doubt the existence of particles, one would have to conclude that the projectile contains "crack particles" whose number is a function of the energy of the projectile – just as the number of photons in a laser beam is a function of its energy, and the number of events produced by a laser beam hitting a photodetector provides information on the impactor's brightness. Only the details are different.

Therefore the number of discrete detection events cannot be regarded as obvious evidence

²⁶This was already clearly expressed in 1924 by JEANS [27], who writes on p. 80: "The fundamental law of quantum-dynamics, that radiant energy is emitted and absorbed only in complete quanta, is no longer interpreted as meaning that the ether can carry radiant energy only in complete quanta, but that matter can deliver or absorb radiant energy only by complete quanta."

for the existence of the same number of associated invisible objects. They are at best evidence of the impact of something.

How do we know whether the tracks in a bubble chamber do not have a similar origin? In both cases (bullet tracks and tracks in a bubble chamber), something impinging on the detector produces a collection of lines or curves. While the details are different, the mechanism is the same. In each case one has a macroscopic and very complicated process that breaks the symmetry and produces tracklike events. Thus there is no a priori reason why in one case but not the other the lines should be interpreted as evidence of particles.

It is strange that tradition says that in the classical experiment with the bullet we see broken symmetry due to microscopic uncertainty, whereas in a bubble chamber we see irreducible quantum randomness.

Tracks in a bubble chamber are also a manifestation of broken symmetry when a radially symmetric alpha-particle field produced by a radioactive nucleus impacts a bubble chamber. A famous paper by MOTT [36] (see also FIGARI & TETA [21, 15]) explains in detail how in a bubble chamber complete particle tracks appear in random directions because of the discrete quantum nature of the bubble chamber – nowhere is made use of the particle nature of the impacting radial wave! Thus while the details are quantum mechanical, the underlying principle is classical!

What we see in a bubble chamber are droplets condensing due to ionization caused by a local piece of a spherical wave emanating from a radioactive nucleus. Mott analyzes the impact of the spherical wave and proceeds without reference to anything outside the quantum formalism. He shows (p.80) that, in the absence of a deflecting magnetic field, the atoms cannot both be ionized unless they lie in a nearly straight line with the radioactive nucleus. There is no direct reference to the α particle causing the ionizations.

Mott's analysis suggests that after the collision the scattered part forms a spherical wave (and not particles flying in different directions) until the wave reaches the detector. The spherical wave is nowhere replaced by flying particles. This makes his analysis very close to a field theoretical treatment. Particles appear to be ghostlike and only macroscopic (hence field-like) things are observed.

Mott needs Born's rule only for interpreting the final outcome in terms of probabilities and finds it consistent with a distribution of straight path only. This fully explains the tracks, without making any claims about position measurements or particle pointer states or collapse assumptions.

Everything said in this and the preceding subsection supports the view that, except during the detection event, particles are unreal in the sense of having no associated beables, and that they have a shade of reality only as identical realizations of a population.

This conclusion that particles are unreal and have meaning only during measurement or

figuratively as part of a population is also reflected in the statistical interpretation of quantum mechanics championed by BALLENTINE [4, Chapter 9], who denies that a single system has a state, and instead assigns the state to a population of similarly prepared systems: He writes on p.240: *"regard the state operator ρ as the fundamental description of the state generated by the thermal emission process, which yields a population of systems each of which is a single electron"*. Thus he gives reality (beable properties) to the preparation procedure, – to the beam –, but not to the electrons (which do not even have a state).

Slightly more indirectly, this is also reflected in the Copenhagen interpretation of quantum mechanics, some versions of which say²⁷ that an unmeasured system has no position, hence (in terms of beables) is unreal.

We conclude that the lack of reality of quantum particles is well supported by the literature, in different ways. In contrast to (F) from Subsection 4.3, and in view of the Copenhagen interpretation, we may rephrase our findings as follows:

(P) Particles do not exist except when measured. They are detection events created by the detector and mediated by fields.

What exists are the beams, and the discrete effects they produce when subjected to measurement. This is an intuitive picture completely orthogonal to the traditional interpretations of quantum mechanics.

It is a historical accident that one continues to use the name particle in the many microscopic situations where it is grossly inappropriate if one thinks of it with the classical meaning of a tiny bullet moving through space. Restrict the use of the particle concept to where it is appropriate, or do not think of particles as "objects" – in both cases all mystery is gone, and the foundations become fully rational.

5 Outlook: New foundations

In the present paper, the universally accepted formal core of quantum physics was cleanly separated from the controversial interpretation issues. Moreover, it was shown that Born's rule, usually considered a fundamental, exact property of nature, has – like most other law of nature – its limitations, and cannot be regarded as a fundamental law.

Based on this insight, the other parts of this series of papers present a new view of the

²⁷This goes back to HEISENBERG[23], who states on p.176: *"However, a single photon of such light is enough to eject the electron completely from its 'path' (so that only a single point of such a path can be defined). Therefore here the word 'path' has no definable meaning."* (*"Von solchem Licht aber genügt ein einziges Lichtquant, um das Elektron völlig aus seiner 'Bahn' zu werfen (weshalb von einer solchen Bahn immer nur ein einziger Raumpunkt definiert werden kann), das Wort 'Bahn' hat hier also keinen vernünftigen Sinn."*) And on p.185 he states that *the 'path' comes into being only when we observe it* (*"Die 'Bahn' entsteht erst dadurch, daß wir sie beobachten"*).

foundations for quantum mechanics (QM) and quantum field theory (QFT). This section puts the results of Parts II–V [40, 41, 42, 43] into perspective, pointing to a coherent quantum physics where classical and quantum thinking live peacefully side by side and jointly fertilize the intuition.

5.1 The thermal interpretation

The insight that Born’s rule has its limitations and hence cannot be the foundation of quantum physics opens the way for an alternative interpretation – the thermal interpretation of quantum physics, defined and discussed in detail in Part II [40], Part III [41], and Part IV [42]. It gives new foundations that connect quantum physics (including quantum mechanics, statistical mechanics, quantum field theory and their applications) to experiment.

The new foundations improve the traditional foundations in several respects:

- The thermal interpretation is independent of the measurement problem. The latter becomes a precise problem in statistical mechanics rather than a fuzzy and problematic notion in the foundations.
- The thermal interpretation better reflects the actual practice of quantum physics, especially regarding its macroscopic implications.
- The thermal interpretation explains the peculiar features of the Copenhagen interpretation (lacking realism between measurements) and the statistical interpretation (lacking realism for the single case) in the microscopic world where the latter apply.
- The thermal interpretation explains the emergence of probabilities from the linear, deterministic dynamics of quantum states or quantum observables.
- The thermal interpretation gives a fair account of the interpretational differences between quantum mechanics and quantum field theory.

5.2 Questioning the traditional foundations in other respects

Traditionally, those learning quantum theory are expected to abandon classical thinking and to learn thinking in a quantum mechanical framework completely different from that of classical mechanics. Though students widely differ in the order in which this happens, sooner or later, most of them are introduced to the items mentioned in the following caricature.

- Typically, they are first told that the Bohr–Sommerfeld theory of quantization gave (for that time) an exact explanation of the spectral lines for the hydrogen atom, firmly establishing that Nature is quantized.

- Then they are told that Bohr's view is obsolete, and that it was just a happy (or even misleading) coincidence that the old quantum theory worked for hydrogen.
- Therefore, they are next acquainted with wave functions on configuration space, their inner product, and the resulting Hilbert spaces of square integrable wave functions. But almost immediately, unnormalizable wave functions are used that do not belong to the Hilbert space.
- They are then told with little intuitive guidance (except for a vague postulated correspondence principle that cannot be made to work in many cases) that in quantum mechanics, observables are replaced by Hermitian operators on this Hilbert space.
- Later they may (or may not) learn that many of these operators are not even defined on the Hilbert space but only on a subspace.
- In particular, they are told that particles have no definite position or momentum, but that these miraculously get random values when measured, due to a postulated collapse of the wave function that prepares the system in a new pure position or momentum state (which does not exist since it is unnormalized).
- Now they must swallow a mysterious law defining the distribution of these random values, called Born's rule. It is justified by the remark that it is proved by (the Stern–Gerlach) experiment. But Born's rule is claimed to hold for all conceivable quantum measurements, although this experiment neither demonstrates the measurement of position nor of momentum or other important quantities.
- They must then learn that between measurements, position and momentum and hence well-defined paths do not exist. This leaves unexplained how the Stern–Gerlach screen can possibly find out that a particle arrived to be measured. The Stern–Gerlach device and all of quantum mechanics begins to look like magic.
- Then they are taught the connection to classical physics by establishing the Ehrenfest theorem for expectation values that obey approximately classical laws. Although well-defined paths do not exist, the system miraculously has at all times a well-defined mean path, even when not measured.
- As a result, classical and quantum physics appear like totally separated realms with totally different concepts and tools, connected only by a rough-and-ready notion of correspondence that is ambiguous and never made precise but works in a few key cases (and always with liberally enough usage).
- After considerable time, when they have some experience with spectral calculations, they learn how to use group theory (or, for those with only little algebra background, spherical harmonics – rotation group representation tools in disguise) to determine the spectrum for hydrogen. Miraculously, the results are identical with those obtained by Bohr.

- At a far later stage, they meet (if at all) coherent states for describing laser light, or as a tool for a semiclassical understanding of the harmonic oscillator. Miraculously, a coherent state happens to perform under the quantum dynamics exact classical oscillations.
- Only few students will also meet coherent states for the hydrogen atom, the Berry phase, Maslov indices, and the accompanying theory of geometric quantization, which gives the (slightly corrected) Bohr–Sommerfeld rules for the spectrum a very respectable place in the quantum theory of exactly solvable systems, even today relevant for semiclassical approximations.
- Even fewer students realize that this implies that, after all, classical mechanics and quantum mechanics are not that far apart. A development of quantum mechanics emphasizing the closeness of classical mechanics and quantum mechanics can be found in my online book, NEUMAIER & WESTRA [44].

Why does the conventional curriculum lead to such a strange state of affairs? Perhaps this is the case because tradition builds the quantum edifice on a time-honored foundation which accounts for essentially all experimental facts but takes a "shut-up-and-calculate" attitude with respect to the interpretation of the foundations. The traditional presentation of quantum physics is clearly adequate for prediction but seems not to be suitable for an adequate understanding.

5.3 Coherent quantum physics

Are coherent states the natural language of quantum theory?

John Klauder, 1986 [30]

The conundrums of Subsection 5.2 are settled in Part V [43] through the development of the concept of a coherent quantum physics that removes the radical split between classical mechanics and quantum mechanics.

That this might be feasible is already suggested by the history of coherent states. (For the early history of coherent states see, e.g., NIETO [47].) Already in 1926, at the very beginning of modern quantum physics, coherent states were used by SCHRÖDINGER [57] to demonstrate the closeness of classical and quantum mechanical descriptions of a physical system. Schrödinger discussed the main properties of the coherent states today known as Glauber coherent states. He did not call them coherent states, a notion coined in 1963 by GLAUBER [22]. Today the term **coherent state** denotes a variety of (in detail very different) collections of states displaying simultaneously a classical and a quantum character.

Part V is also part of another series of papers (see NEUMAIER [38]) that develop a theory of coherent spaces, in terms of which the notion of a Glauber coherent state is further generalized. Together with the thermal interpretation from Part II, it serves as the basis for new foundations for quantum physics in which classical and quantum thinking live peacefully side by side and jointly fertilize the intuition. The fundamental importance of coherent states is emphasized by defining a **coherent quantum physics**, based on the concept of coherence in various forms, thereby rebuilding from scratch the foundations of quantum physics. Summarizing the vision in the shortest terms, we may say:

- Coherent quantum physics is quantum physics in terms of a coherent space consisting of a line bundle over a classical phase space and an appropriate "coherent product" characterizing the physical properties of a quantum system.
- The kinematical structure of quantum physics and the meaning of the q-observables are given by the symmetries of this coherent space.
- The dynamics is given by von Neumann's equation for the density operator.
- The connection to experiment is given through the thermal interpretation.

Coherent spaces reconcile the old (semiclassical, Bohr-style) thinking with the requirements of the new (operator-base) quantum physics. They become the foundation on which a better, coherent quantum physics is built. Mathematically, these foundations are equivalent to the traditional Hilbert space approach. But conceptually, these foundations begin with what is common between the classical and the quantum world.

The only miracles left in the new approach outlined in this paper are of a linguistic nature, namely the coincidence that the same word "coherence" fits several different contexts that come together in the new foundations:

- spatio-temporal coherence, the meaning of the term in "coherent states",
- logical coherence, referring to mathematically sound foundations,
- intuitive coherence, implying that concepts make holistic sense to the intellect, and
- coherence as harmony, the meaning of the term in "coherent configurations" and "coherent algebras", concepts from the combinatorics of symmetry.²⁸

²⁸Coherent configurations and the associated coherent algebras were introduced in pure mathematics around 1970 (by HIGMAN [25, 26]), completely independent of physical considerations. Special classes of coherent configurations called **association schemes** and **distance-regular graphs** are very well-studied, and many interesting examples are known in detail.

Coherent configurations are, in a sense made precise in the companion paper [39], closely related to a finite variant of coherent states. Just as semisimple Lie groups act as symmetry groups of associated Riemannian symmetric spaces, and their representation theory leads to Perelomov coherent states, so most finite simple groups act as symmetry groups of associated distance regular graphs. The latter is recorded in the book by BROUWER et al. [11]), of which I am a coauthor. The book appeared just about the time when, for completely different reasons, I turned to seriously study quantum physics.

Only much later, I realized the extent of the connection of that work to coherent states. The two subjects

In the literature, one usually finds coherent states discussed just for themselves, or in the context of the classical limit. However, they are also a powerful instrument in other respects. The reason is that they have both a good intuitive semiclassical interpretation and give good access to the whole Hilbert space (and beyond). Indeed, coherent quantum physics turns coherent states into the fundamental tool for studying quantum physics.

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