

Foundations of quantum physics

IV. More on the thermal interpretation

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Abstract. This paper continues the discussion of the thermal interpretation of quantum physics. While Part II and Part III of this series of papers explained and justified the reasons for the departure from tradition, the present Part IV summarizes the main features and adds intuitive explanations and new technical developments.

It is shown how the spectral features of quantum systems and an approximate classical dynamics arise under appropriate conditions.

Evidence is given for how, in the thermal interpretation, the measurement of a qubit by a pointer q-expectation may result in a binary detection event with probabilities given by the diagonal entries of the reduced density matrix of the prepared qubit.

Differences in the conventions about measurement errors in the thermal interpretation and in traditional interpretations are discussed in detail.

Several standard experiments, the double slit, Stern–Gerlach, and particle decay are described from the perspective of the thermal interpretation.

For the discussion of questions related to this paper, please use the discussion forum <https://www.physicsoverflow.org>.

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1 Introduction

This paper, the fourth of the series on the foundations of quantum physics, continues the discussion of the thermal interpretation of quantum physics by summarizing the main features, by adding intuitive explanations and by introducing new technical developments.

In this introductory section, we first show that the notion of indistinguishability of the individual constituents of a system naturally leads to the thermal interpretation view. We then summarize the most important points from the thermal interpretation, discussed in detail in Part II [12] and applied to measurement in Part III [13] of this series of papers.

Section 2 shows how the spectral features of quantum systems and an approximate classical dynamics arise under appropriate conditions.

In Section 3, formal evidence is given for how, in the thermal interpretation, the measurement of a qubit by a pointer q-expectation may result in a binary detection event with probabilities given by the diagonal entries of the reduced density matrix of the prepared qubit.

In the thermal interpretation, the true properties of a quantum system, approximately revealed by a measurement, are the q-expectations rather than the eigenvalues. After nearly a century of conditioning to the opposite convention specified in Born's rule, this radical change of interpretation seems at first sight very counterintuitive. A detailed justification and comparison with the traditional convention is given in Section 4 from the measurement point of view. An analysis of the double slit experiment leads to the picture of a quantum bucket for measuring a continuous variable with a device capable only of producing discrete results.

The final Section 5 shows how the notion of quantum currents may be used to visualize in the thermal interpretation the finite time dynamics of particle decay and the Stern–Gerlach experiment. In the case of observing angular momentum, the measurement process is claimed to systematically introduce $O(\hbar)$ perturbations of the same kind as rounding errors in floating-point computations – a tiny amount for all but microscopic measurements. Due to the representation theory of the compact rotation group, these discretize the response of the measurement device to the continuous signal represented by the q-expectation $\langle J \rangle$ of the vector-valued angular momentum J of the measured particle. This results (depending on the precision of the measuring device) in almost¹ exact multiples of $\frac{1}{2}\hbar$, with resulting discretization errors of order $O(\hbar)$. For a single electron spin measured in a Stern–Gerlach experiment, this perturbation is of the same order as the size of each component of $\langle J \rangle$,

¹In the past, this experiment (and others) could be used for precision measurements of \hbar . But from May 20, 2019 onwards, \hbar has by convention a fixed (but irrational) value, as part of the 2019 redefinition of SI base units [1]. From then on, one can get (by calibration) exact multiples of $\frac{1}{2}\hbar$, as claimed in Born's rule. However, the thermal interpretation asserts that, since the measurement results are not reproducible, this seeming exactness of the angular momentum measurement is a spurious artifact of measuring it with a quantum bucket.

which is bounded itself by $\frac{1}{2}\hbar$, resulting – in the convention of the thermal interpretation – in intrinsic measurement errors of up to 200 percent. But there is no logical problem since any number of order $O(\hbar)$ is an $O(\hbar)$ perturbations of any other number of order $O(\hbar)$.

In the bibliography, the number(s) after each reference give the page number(s) where it is cited.

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1.1 Properties of anonymous collections

Let x_k ($k = 1, \dots, N$) denote the (real) values of some property of a collection of N similar classical objects. If the detailed identification of the objects is deemed irrelevant for certain purposes, the assignment of indices to the the individual objects may be dropped, in this way anonymizing the data. Indeed, this is a common procedure in the statistical practice of handling sensitive data. Once this is done we can no longer say which property belongs to which object – in the resulting description, the objects have become anonymous, or **indistinguishable**.

As a consequence, the individual values x_k play no longer a useful role in the anonymized collection. From a mathematical point of view, only symmetric functions of the x_k retain meaningful information about the collection. By a well-known theorem, every symmetric polynomial (and by taking limits, therefore any symmetric analytic function) of the x_k can be written as a function of the power sums $\sum x_k^e$ ($e = 1, 2, 3, \dots$), equivalently, as a function of the sample expectations $\langle x^e \rangle = N^{-1} \sum_{k=1}^N x_k^e$. Some discontinuous symmetric

functions also play a role, and can be written as a function of sample expectations of discontinuous functions. Thus all meaningful properties of the anonymized collection are encoded in expectations $\langle f(x) \rangle$ of functions of the anonymous value x of an anonymous object of the collection. These expectations may therefore be regarded as the **beables** of the anonymous classical collection.

It is precisely this situation that probability theory and statistics cater for – the description of anonymous events, not that of actual events! We assign probabilities to anonymous events such as "casting a die gives a six" (where the indefinite article indicates an anonymous die), not to the number of eyes shown on a particular die cast at a particular time (which is not a random variable but a fixed, though possibly unknown value). We estimate the expected lifetime of "a 45 year old French male", not that of Francois Renon from Calais, say. And so on. Formally, from what is mathematically modeled, anonymous objects (whose only

properties are expectation values and probabilities) are very different from **typical** objects, which are identifiable examples of particular objects (whose properties are individual values within observable typical ranges).

Even before the advent of quantum mechanics, it turned out that, in classical statistical mechanics, atoms are indistinguishable not only due to practical limitations but in principle, and that there is no theoretically conceivable way to distinguish them as individuals – if it were possible, the resulting predictions would have an additional entropy of mixing, which is in conflict with the observed thermodynamical properties of bulk systems. This means that there are fundamental constraints that forbid the atoms in a classical multiparticle system to have individual properties. Thus, in a classical multiparticle system, the atoms are anonymous objects without an identity, and the expectations are the only classical beables. This directly leads to the main innovation of the thermal interpretation – that expectations are beables.

This situation persists in the quantum case, where atoms and elementary particles are in principle² indistinguishable, too. Thus the atoms and elementary particles in a multiparticle quantum system are also anonymous objects without an identity. This is reflected in the fact that on the physical Hilbert space of correctly symmetrized wave functions, no particle position operator is definable; particle positions are spurious objects. The definable operators are cumulative N -particle operators. When expressed in terms of the second quantization formalism, these become quantum fields. Thus the q-expectations of quantum fields are the natural generalizations of the classical beables.

1.2 Thermal interpretation summary

The thermal interpretation treats quantum physics in a deterministic, almost classical fashion. It differentiates between

- **properties** of quantum systems – beables, that the systems possess according to a scientific model, independent of whether these properties are known or even knowable), and
- **experiments** consisting of a sequence of **measurements** – which are the scientists’s approximate way of checking such properties and validating the corresponding models.

In classical mechanics, particles exist. States define their properties (the beables of classical mechanics), which are given by the exact positions and momenta of the particles, some of which can be approximately measured. From a fundamental point of view, fields are (as

²Exceptions are cases where the range of some quantity identifies a unique particle. (This is analogous to the identifiability of outliers in anonymous statistical data.) Examples include a single atom prepared in an ion trap, single atoms on the surface of some other material, the atom closest to a given lattice position in a piece of metal, or, in the Hartree-Fock approximation, the outermost electron of an atom. In this case, the identification can be made by expectations of quantities containing a characteristic function of the defining property as a factor.

in classical continuum mechanics) only coarse-grained approximate concepts. This is the standard interpretation of classical physics.

In quantum field theory, fields exist. States define their properties (the beables of quantum field theory), which are given by the exact q -expectations of the fields and their appropriately normally ordered or time-ordered products, some of which can be approximately measured. From a fundamental point of view, particles are (as in classical geometric optics) only coarse-grained approximate concepts. This is the thermal interpretation of quantum physics.

Experimental physics is in both cases about how to do the measurements, and under which conditions which measurements are how accurate. This is achieved using the standard theory based upon three ingredients: the formal core of quantum mechanics, the respective foundations, and

(CC) Callen's criterion (cf. CALLEN [2, p.15]): Operationally, a system is in a given state if its properties are consistently described by the theory for this state.

The thermal interpretation makes quantum physics as deterministic as classical physics, and explains all random quantum effects as resulting from coarse-graining, as in classical physics. Quantized measurement results (as observed angular momentum measurements) are explained by environment-induced randomness and environment-induced dissipation, as for a classical, environment-induced diffusion process in a double-well potential. Born's statistical interpretation follows, in the limited range where it applies, from this and the deterministic rules.

The deterministic Ehrenfest dynamics of the collection of all q -expectations couples local q -expectations (e.g., idealized pointer readings) to multilocal q -expectations, and accounts in this way for the nonclassical correlations observed in long-distance entanglement.

A subsystem of a composite system is selected by picking a vector space of quantities (linear operators) relevant to the subsystem. The existence of multilocal q -expectations implies that a composite system is more than its parts. Regarding a tensor product of two systems as two separate subsystems (as often done informally) is appropriate only when all quantities that correlate the two systems are deemed irrelevant. If this is not the case, thinking of the composite system only in terms of its subsystems produces the weird features visible in many discussions of quantum entanglement.

1.3 Advantages of the thermal interpretation

[...] auf die objektive Beschreibbarkeit der individuellen Makro-Systeme (Beschreibung des 'Realzustandes') nicht verzichtet werden kann ohne dass das physikalische Weltbild gewissermassen sich in einen Nebel auflöst. Schliesslich ist die Auffassung wohl unvermeidbar, dass die Physik nach einer Realbeschreibung des Einzel-Systems streben muss. Die Natur als Ganzes kann eben nur als individuelles (einmalig existierendes) System gedacht werden und nicht als eine 'System-Gesamtheit'.

Albert Einstein, 1953 [4, p.40]

The conventions embodied in the thermal interpretation have, compared to tradition, several direct or indirect advantages.

- The thermal interpretation allows a consistent quantum description of the universe and its subsystems, from the smallest to the largest levels of modeling, including its classical aspects.
- The thermal interpretation preserves the agreement of quantum theory with the experimental record.
- At the levels of the postulates, the thermal interpretation requires much less technical mathematics (no spectral theorem, no notion of eigenvalue, no probability theory).
- The foundations are easily stated and motivated since they are essentially the foundations used everywhere for uncertainty quantification.
- The thermal interpretation allows one to make definite statements about each single quantum system, no matter how large or small it is.
- The thermal interpretation eliminates from the foundations the philosophically problematic notions of probability and measurement.

As a result of the multi-valuedness of the true values, Born's statistical interpretation needs probabilities in the very foundations of quantum physics. In contrast, in the thermal interpretation, probabilities are absent in the foundations of quantum physics, as a result of the single-valuedness of the true values.

Every observable quantity A has an associated intrinsic state-dependent uncertainty σ_A within which it can be determined (in principle). According to the thermal interpretation it is as meaningless to ask for more accuracy as to ask for the position of an apple to mm accuracy. Statistics enters whenever a single value has too much uncertainty, and only then. In this case, the uncertainty can be reduced by calculating means, as within classical physics.

- Position and momentum of distinguishable particles have at any time simultaneously idealized but uncertain values (just like the position and momentum of a classical rocket),

eliminating the spooky nature of the traditional quantum ontologies.

- The thermal interpretation solves the measurement problem, makes quantum mechanics much less mysterious, and makes it much less different from classical mechanics.

That quantities with large relative uncertainty (such as single spins) are erratic in measurement is nothing special to quantum physics but very familiar from the measurement of classical noisy systems. The thermal interpretation asserts (and gives good grounds for trusting) that all uncertainty is of this kind, and probabilities enter only at the same level as in classical physics – as residual uncertainty of approximate, coarse-grained treatments.

- Open problems concerning technical details (mentioned at the end of Part III) provide mathematical challenges, and show that, unlike traditional interpretations, the thermal interpretation is in principle refutable by theoretical arguments.

2 Classical and spectral features of quantum physics

Es ist wünschenswert, die folgende Frage möglichst elementar beantworten zu können: Welcher Rückblick ergibt sich vom Standpunkt der Quantenmechanik auf die Newtonschen Grundgleichungen der klassischen Mechanik?

Paul Ehrenfest 1927 [6]

In 1927, when the Copenhagen interpretation (the informal agreement on the interpretation reached at the 1927 Como and Solvay conferences) was forged, its main purpose was to reconcile the then new quantum formalism with the experimental evidence available at that time. Apart from the Stern–Gerlach experiment, the evidence consisted exclusively of (i) the observation of spectra of atoms and molecules, and (ii) the need to reconcile the quantum description of the invisible microscopic details with the classical description of the macroscopic world.

This section shows that the same evidence is naturally explained by the thermal interpretation. Indeed, with a little more work and imagination, Paul EHRENFEST, whose paper [6] appeared in 1927, could have easily found and justified this interpretation.

2.1 The classical approximation

We consider an interacting multiparticle quantum system with mass matrix M , position operator q , and momentum operator p , with dynamics given by the Hamiltonian $H = \frac{1}{2}p^T M^{-1}p + V(q)$. To arrive at an approximate classical equation of motion for the q -expectation $\bar{q} = \langle q \rangle$, we apply the Ehrenfest equation and find, using the canonical

commutation relations and componentwise expectations, the formulas

$$\frac{d}{dt}\langle q \rangle = M^{-1}\langle p \rangle, \quad \frac{d}{dt}\langle p \rangle = -\langle \nabla V(q) \rangle,$$

hence the equation

$$M \frac{d^2}{dt^2} \bar{q} + \langle \nabla V(q) \rangle = 0. \quad (1)$$

by EHRENFEST [6], who observed the close formal relationship with the classical equation of motion

$$M \frac{d^2}{dt^2} q + \nabla V(q) = 0 \quad (2)$$

for this Hamiltonian. To turn this formal relationship into a quantitative approximation, we first prove the following

Approximation Lemma. Let f be a twice continuously differentiable complex-valued function on \mathbb{R}^n . Then, for every vector q of n commuting self-adjoint quantities with convex joint spectrum and every state, we have (with the spectral norm)

$$|f(q) - f(\bar{q})| \leq \frac{1}{2} \|f''(q)\| \sum_{k=1}^n \sigma_{q_k}^2. \quad (3)$$

Indeed, for any \tilde{q} in the joint spectrum of q and $\varepsilon = \tilde{q} - \bar{q}$, we have

$$f(\tilde{q}) = f(\bar{q} + \varepsilon) = f(\bar{q}) + f'(\bar{q})\varepsilon + \int_0^1 \varepsilon^T f''(\bar{q} + s\varepsilon) \varepsilon ds.$$

By assumption, $\bar{q} + s\varepsilon$ is for all $s \in [0, 1]$ in the joint spectrum of q , hence by definition of the spectral norm,

$$\left| \frac{\varepsilon^T f''(\bar{q} + s\varepsilon) \varepsilon}{\varepsilon^T \varepsilon} \right| \leq \|f''(\bar{q} + s\varepsilon)\|_2 \leq \|f''(q)\|.$$

Therefore

$$\begin{aligned} \left| f(\tilde{q}) - f(\bar{q}) - f'(\bar{q})(\tilde{q} - \bar{q}) \right| &\leq \int_0^1 \|f''(q)\| \varepsilon^T \varepsilon ds \\ &= \frac{1}{2} \|f''(q)\| \varepsilon^T \varepsilon = \frac{1}{2} \|f''(q)\| \sum_{k=1}^n (\tilde{q}_k - \bar{q}_k)^2, \end{aligned}$$

This inequality therefore also holds for q in place of \tilde{q} . Taking q -expectations, we find

$$\begin{aligned} |\langle f(q) - f(\bar{q}) \rangle| &= \left| \left\langle f(q) - f(\bar{q}) - f'(\bar{q})(q - \bar{q}) \right\rangle \right| \\ &\leq \frac{1}{2} \|f''(q)\| \sum_{k=1}^n \left\langle (q_k - \bar{q}_k)^2 \right\rangle = \frac{1}{2} \|f''(q)\| \sum_{k=1}^n \sigma_{q_k}^2, \end{aligned}$$

proving the lemma. Returning to our original goal, we rewrite (1) in the form

$$M \frac{d^2}{dt^2} \bar{q} + \nabla V(\bar{q}) = -\langle \nabla V(q) - \nabla V(\bar{q}) \rangle$$

and apply the approximation lemma to the right hand side. Under the assumption that the potential V is three times continuously differentiable and the spectrum of the third derivative $V'''(q)$ is bounded by a constant C , we may conclude the differential inequality

$$\left| M \frac{d^2}{dt^2} \bar{q} + \nabla V(\bar{q}) \right| \leq C \sum_{k=1}^n \sigma_{q_k}^2.$$

Thus as long as the uncertainties σ_{q_k} remain sufficiently small, the classical dynamical law (2) holds with good accuracy for the q -expectation \bar{q} in place of q .

Under these conditions, which hold by the weak law of large numbers whenever q refers to the center of mass coordinates of macroscopic spherical bodies at macroscopic distances from each other, the q -expectations satisfy the traditional classical equation of motion. This proves that Newton's mechanics is a macroscopic approximation to the quantum dynamics of q -expectations.

Thus classical physics emerges without any difficulty from the thermal interpretation together with the weak law of large numbers.

In a similar way, one can justify the quantum classical dynamical models discussed in Subsection 4.6 of Part III [13], which treat only some low uncertainty quantities as classical and keep the quantum nature of the remaining ones.

2.2 The Rydberg–Ritz combination principle

Here we show that in any quantum system, the differences of the energy levels (the eigenvalues of the Hamiltonian H) are in principle directly observable, since they represent excitable oscillation frequencies of the system and thus can be probed by coupling the system to a harmonic oscillator with adjustable frequency. Thus the observed spectral properties of quantum systems appear in the thermal interpretation as natural resonance phenomena.

To see this, we shall assume for simplicity a quantum system whose Hamiltonian has a purely discrete spectrum. For a partially continuous spectrum, analogous results, in which sums are replaced by Stieltjes integrals, can be proved using the Gel'fand–Maurin theorem, also known under the name nuclear spectral theorem (cf. MAURIN [9]).

We work in the Heisenberg picture in a basis of eigenstates of the Hamiltonian, such that $H|k\rangle = E_k|k\rangle$ for certain energy levels E_k . The q -expectation

$$\langle A(t) \rangle = \text{Tr } \rho A(t) = \sum_{j,k} \rho_{jk} A_{kj}(t)$$

is a linear combination of the matrix elements

$$A_{kj}(t) = \langle k|A(t)|j\rangle = \langle k|e^{iHt/\hbar} A e^{-iHt/\hbar}|j\rangle = e^{iE_k t/\hbar} \langle k|A|j\rangle e^{-iE_j t/\hbar} = e^{i\omega_{kj}t} \langle k|A|j\rangle,$$

where

$$\omega_{kj} = \frac{E_k - E_j}{\hbar}. \quad (4)$$

Thus the q-expectation exhibits multiply periodic oscillatory behavior whose frequencies ω_{jk} are scaled differences of energy levels. This relation, the modern form of the **Rydberg–Ritz combination principle** found in 1908 by RITZ [17], may be expressed in Planck’s form³

$$\Delta E = \hbar\omega. \quad (5)$$

To probe the spectrum of a quantum system, we bring it into contact with a macroscopically observable (hence classically modeled) weakly damped harmonic oscillator. For simplicity we treat just a single harmonic oscillator. In practice, one often observes many oscillators simultaneously, e.g., by observing the oscillations of the electromagnetic field in the form of electromagnetic radiation – light, X-rays, or microwaves. However, in most cases the oscillators may be regarded as independent and noninteracting. The result of probing a system with multiple oscillators results in a linear superposition of the results of probing with a single oscillator. This is a special case of the general fact that solutions of linear differential equations depend linearly on the right hand side.

From the point of view of the macroscopically observable classical oscillator, the probed quantum system appears simply as a time-dependent external force $F(t)$ that modifies the dynamics of the free harmonic oscillator. Instead of the harmonic equation $m\ddot{q} + c\dot{q} + kq = 0$ with real $m, c, k > 0$, we get the differential equation describing the **forced harmonic oscillator**, given by

$$m\ddot{q} + c\dot{q} + kq = F(t),$$

where the external force F is the q-expectation

$$F(t) = \langle A(t) \rangle$$

of a quantity A from the probed system. We assume the oscillator to have an adjustable frequency

$$\omega = \sqrt{\frac{k}{m}} > 0$$

and consider the response as a function of ω at fixed mass m and stiffness $k = m^2\omega$.

³The formula (5) appears first in the famous 1900 paper by PLANCK [16] on the radiation spectrum of a black body. Planck wrote it in the form $\Delta E = h\nu$, where $h = 2\pi\hbar$ and $\nu = \omega/2\pi$ is the linear frequency. The symbol for the quotient $\hbar = h/2\pi$, which translates this into our formula was invented much later, in the 1930 quantum mechanics book by DIRAC [3].

If the measurement is done far from the probed system, such as a measurement of light (electromagnetic radiation) emitted by a far away source (e.g., a star, but also a Bunsen flame observed by the eye), the back reaction of the classical oscillator on the probed system can be neglected. Then the probed system can be considered as isolated and evolves according to the preceding analysis, hence the external force F can be written as a superposition

$$F(t) = \sum_l F_l e^{i\omega_l t},$$

of exponentials oscillating with the (positive and negative) Rydberg–Ritz frequencies, rearranged in linear order. The solution to the differential equation consists of a particular solution and a solution to the homogeneous equation. Due to damping, the latter is transient and decays to zero. There is a distinguished particular solution persisting after the transient decayed, which oscillates with the same frequencies as the force, easily seen to be given by

$$q(t) = \sum_l q_l e^{i\omega_l t}, \quad q_l = \frac{F_l}{m(\omega^2 - \omega_l^2) + ic\omega_l}.$$

Since the frequencies are real and distinct, the denominator cannot vanish. The energy in the l th mode is therefore proportional to the amplitude

$$|q_l|^2 = \frac{|F_l|^2}{m^2(\omega^2 k - \omega_l^2)^2 + c^2 \omega_l^2}, \quad (6)$$

with a maximum at the resonance frequency $\omega = |\omega_l|$. The total energy is proportional to

$$|q(t)|^2 = \sum_l |q_l|^2 + \sum_{k \neq l} q_k^* q_l e^{i(\omega_k - \omega_l)t}. \quad (7)$$

We now look at the short-time average (recorded by a typical detector). If the frequencies ω_k with significant intensity are well-separated, the oscillating terms in (7) cancel out and we find a total mean energy proportional to

$$a(t) \approx \sum_l |q_l|^2 = \sum_l \frac{|F_l|^2}{(m^2(\omega^2 k - \omega_l^2)^2 + c^2 \omega_l^2)}.$$

As a function of the varying frequency, this has the typical spectral intensity form of a superposition of Lorentz shaped resonance curves, with local maxima very close to the resonance frequencies $|\omega_l|$.

3 Measuring single qubits

In this section we consider in detail how the thermal interpretation explains the emergence of binary responses of a measurement device when coupled with the simplest quantum

object, a qubit, with probabilities given by the diagonal entries of the reduced density matrix of the prepared qubit.

3.1 Physical systems and their states

From a fundamental point of view, each physical system is a subsystem of the whole **universe**, the only truly isolated physical system containing the solar system.

In the standard Schrödinger picture, the universe has at each time t a universal density operator

$$\rho(t) = e^{-itH/\hbar} \rho(0) e^{itH/\hbar},$$

in terms of which the q-expectations $\langle A \rangle_t = \text{Tr } \rho(t) A$ at time t are defined. In the covariant Schrödinger picture introduced in Part II [12], the universe has at each spacetime position x a universal density operator

$$\rho(x) = e^{-ip \cdot x/\hbar} \rho(0) e^{ip \cdot x/\hbar},$$

in terms of which the q-expectations $\langle A \rangle_x = \text{Tr } \rho(x) A$ at spacetime position x are defined.

A **physical system** is a subsystem of the universe. It is selected by distinguishing the elements of a vector space \mathbb{E} of quantities (linear operators on the Hilbert space of the universe) as being the quantities relevant to the subsystem, and restricting the q-expectation mapping of the universe to \mathbb{E} .

In many cases, the physical system S is defined by a decomposition of the Hilbert space \mathbb{H} of the universe into a tensor product $\mathbb{H} = \mathbb{H}^S \otimes \mathbb{H}^E$ of a **system Hilbert space** \mathbb{H}^S and an **environment Hilbert space** \mathbb{H}^E for the remaining part of the universe. We call such physical systems **standard**.

Each standard physical system S has a corresponding reduced density operator, given in the standard Schrödinger picture by $\rho^S(t) := \text{Tr}_E \rho(t)$ and in the covariant Schrödinger picture by $\rho^S(x) := \text{Tr}_E \rho(x)$, where Tr_E denotes the partial trace over the environment. We call the reduced density operators $\rho^S(t)$ and $\rho^S(x)$ the **state** of the physical system at time t or at spacetime position x , respectively. These are *the only* states the thermal interpretation is concerned with at all – because these are the states containing precisely the information about the q-expectations of operators of the universe attached to the system S . Indeed, the reduced density operator is defined such that for linear operators A on \mathbb{H}^S describing system properties, the q-expectations are given by

$$\langle A \rangle_t := \langle A \otimes 1 \rangle_t = \text{Tr } \rho(t) (A \otimes 1) = \text{Tr } \rho^S(t) A,$$

$$\langle A \rangle_x := \langle A \otimes 1 \rangle_x = \text{Tr } \rho(x) (A \otimes 1) = \text{Tr } \rho^S(x) A,$$

where 1 denotes the identity operator on \mathbb{H}^E . Each $\rho^S(t)$ and $\rho^S(x)$ is a Hermitian positive semidefinite linear operator on \mathbb{H}^S with trace 1. Given any Hermitian positive semidefinite linear operator ρ^S on \mathbb{H}^S with trace 1, it may be possible, by utilizing the laws of Nature and the control facilities these impart on humans or machines, to ensure that at some time t_{prep} (or some spacetime position x_{prep}), $\rho^S(t_{\text{prep}})$ resp. $\rho^S(x_{\text{prep}})$ approximates ρ^S sufficiently well that predictions with ρ^S in place of $\rho^S(t_{\text{prep}})$ or $\rho^S(x_{\text{prep}})$ match experimental checks. In this case we say that at time t_{prep} (or spacetime position x_{prep}), the system S is **prepared** in the state ρ^S . How to do this is part of the experimental art of **preparation**.

If ρ^S has rank 1 then $\rho^S = \psi\psi^*$ for some **state vector** ψ of norm one (determined by ρ^S up to a phase). In this case we say that the system is prepared in the **pure state** ψ . Physicists can prepare a system in a pure state only when this system has very few degrees of freedom.

3.2 A single qubit

We consider a single qubit as a subsystem of the universe. The Hilbert space of the universe can be decomposed into a tensor product $\mathbb{H} = \mathbb{H}^S \otimes \mathbb{H}^E$ of a 2-dimensional system Hilbert space \mathbb{H}^S and an environment Hilbert space \mathbb{H}^E for the remaining part of the universe. We suppose that the qubit is prepared in a state defined by a general reduced density matrix ρ^S with components $\rho_{jk}^S = \langle j|\rho^S|k\rangle$. Then ρ^S is given by

$$\rho^S = \sum_{j,k} \rho_{jk}^S |j\rangle\langle k|.$$

Since ρ^S is Hermitian positive semidefinite with trace 1,

$$\rho^S = \begin{pmatrix} p & \alpha^* \\ \alpha & 1-p \end{pmatrix} \quad (8)$$

for some real number $p \in [0, 1]$ and some complex number α with

$$|\alpha| \leq \sqrt{p(1-p)}. \quad (9)$$

According to the thermal interpretation, the true value of the up operator $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is

$$\bar{A} = \langle A \rangle = \text{Tr}_S A = p,$$

with an uncertainty of

$$\sigma_A = \sqrt{\langle A^2 \rangle - \bar{A}^2} = \sqrt{p(1-p)}.$$

In particular, the true value has no intrinsic uncertainty iff $p = 0$ or $p = 1$.

3.3 The response of environmental beables

In the following we analyze in which way p is reflected in an arbitrary environmental q-expectation. For simplicity, we assume that at preparation time $t = 0$, the density operator of the universe in the Schrödinger picture has the tensor product form

$$\rho_0 = \rho^S \otimes \rho^E = \sum_{j,k} \rho_{jk}^S |j\rangle\langle k| \otimes \rho^E. \quad (10)$$

(This assumption could be relaxed but not without going through much more technical computations.) The dynamics of the universe is governed by a unitary matrix $U(t)$ turning ρ_0 into

$$\rho(t) = U(t)\rho_0U(t)^*.$$

We may decompose $U(t)$ uniquely as

$$U(t) = \sum_{\ell,k} |\ell\rangle\langle k| \otimes U_{\ell k}(t)$$

with suitable $U_{\ell k}(t) \in \text{Lin } \mathbb{H}^E$.

Let $X^E \in \text{Lin } \mathbb{H}^E$ be a Hermitian quantity located in the environment, so that

$$X := 1 \otimes X^E \in \text{Lin } \mathbb{H}$$

is a quantity of the universe. We want to calculate its q-expectation

$$\overline{X}_t := \langle X \rangle_t = \text{Tr } \rho(t)X = \text{Tr } U(t)\rho_0U(t)^*X = \text{Tr } \rho_0U(t)^*XU(t) = \text{Tr } \rho_0X(t),$$

where

$$\begin{aligned} X(t) &= U(t)^*XU(t) = U(t)^*(1 \otimes X^E)U(t) \\ &= \sum_{\ell,j} |j\rangle\langle \ell| \otimes U_{\ell j}(t)^*(1 \otimes X^E) \sum_{\ell',k} |\ell'\rangle\langle k| \otimes U_{\ell' k}(t) \\ &= \sum_{\ell,\ell',j,k} |j\rangle\langle \ell|\ell'\rangle\langle k| \otimes U_{\ell j}(t)^*X^E U_{\ell' k}(t) \\ &= \sum_{\ell,j,k} |j\rangle\langle k| \otimes U_{\ell j}(t)^*X^E U_{\ell k}(t) \end{aligned}$$

Using (10), we find that

$$\begin{aligned} \overline{X}_t &= \langle X \rangle_t = \text{Tr } \rho_0X(t) \\ &= \text{Tr} \sum_{j',k'} \rho_{j'k'}^S |j'\rangle\langle k'| \otimes \rho^E \sum_{\ell,j,k} |j\rangle\langle k| \otimes U_{\ell j}(t)^*X^E U_{\ell k}(t) \\ &= \sum_{\ell,j,k} \rho_{kj}^S |k\rangle\langle j| \text{Tr}_E \rho^E U_{\ell j}(t)^*X^E U_{\ell k}(t). \end{aligned}$$

If we define $X^S(t) \in \text{Lin } \mathbb{H}^S$ by

$$X^S(t)_{jk} := \text{Tr}_E \rho^E U_{\ell_j}(t)^* X^E U_{\ell_k}(t), \quad (11)$$

we arrive at

$$\bar{X}_t = \sum_{\ell,j,k} \rho_{kj}^S |k\rangle\langle j| X_{jk}^S(t) = \text{Tr}_S \rho^S X^S(t),$$

and by (8),

$$\bar{X}_t = p X_{11}^S(t) + (1-p) X_{22}^S(t) + 2 \text{Re } \alpha X_{12}^S(t). \quad (12)$$

We now consider multiple preparations in the qubit state represented by ρ^S , but in multiple contexts. We label each such preparation with a label ω from some sample space Ω . Since the split into system and environment is different in each preparation, the state ρ^E representing the state of the environment depends on the preparation label ω . Since X^E was assumed to be Hermitian, (11) implies that the matrix $X^S(t)$ is also Hermitian and, being dependent on ρ^E , depends on ω . We write

$$X^S(t) = \begin{pmatrix} \hat{x}_t(\omega) & \hat{z}_t(\omega)^* \\ \hat{z}_t(\omega) & \hat{y}_t(\omega) \end{pmatrix}$$

for the realization obtained in the preparation labelled by $\omega \in \Omega$. Then we may rewrite (12) as

$$\bar{X}_t = p \hat{x}_t(\omega) + (1-p) \hat{y}_t(\omega) + 2 \text{Re } \alpha \hat{z}_t(\omega). \quad (13)$$

The ω -dependence is actually a dependence on details of the environment that are uncontrollable in practice. Hence it effectively turns $X^S(t)$ into a time-dependent random matrix and \hat{x}_t , \hat{y}_t and \hat{z}_t into time-dependent random variables, of which a new realization is obtained for each preparation of the qubit in the state represented by ρ^S . Their distribution, however, depends on more general properties of the environment and is in principle amenable to an analysis by the traditional techniques of statistical mechanics. Let us write

$$X^{\text{eff}}(t) = \begin{pmatrix} x_t^{\text{eff}} & z_t^{\text{eff}*} \\ z_t^{\text{eff}} & y_t \end{pmatrix}$$

for the effective mean of $X^S(t)$, averaged over all preparations $\omega \in \Omega$. As a consequence of (12), \bar{X}_t itself behaves like a random variable, with mean

$$\bar{X}_t^{\text{eff}} = \text{Tr}_S \rho^S X^{\text{eff}}(t) = \langle X^{\text{eff}}(t) \rangle_S.$$

Thus we may view every environmental q-expectation as a randomized observation of a corresponding effective q-expectation of a quantity $X^{\text{eff}}(t)$ defined on the qubit. Usually, $X^{\text{eff}}(t)$ is just noise and \bar{X}_t^{eff} is essentially zero, giving no information about the qubit.

However, for specially chosen X , namely for those where X is physically related to the qubit in a significant way, $\overline{X}_t^{\text{eff}}$ is nonzero and gives nontrivial statistical information about the qubit measured – it is part of a useful measurement device for qubits.

Which qubit quantity is observed can be found out by techniques known from quantum tomography. If X depends on a parameter vector θ , then $X^{\text{eff}}(t)$ also depends on θ , and we can find out the precise θ -dependence by these techniques. Thus we have an effective way of calibrating our measurement device. In particular, whenever we can find a value for θ for which $X^{\text{eff}}(t) = A$ we get a statistical measurement of the true value p of the up operator A .

3.4 The emergence of Born's rule

The precise statistical properties of $X^S(t)$ can be found out by careful calibration. It can also be predicted by a theoretical analysis of the formula defining $X^S(t)$, using the standard techniques of statistical mechanics, though this may involve considerable work. Here we give an outline of how such a theoretical analysis may proceed, leaving details to future investigations of particular situations amenable to a more detailed analysis.

We consider an environmental operator X^E that leads to a **pointer variable**, here a real number \overline{X}_t that moves in a macroscopic time $t > 0$ a macroscopic distance to the left (in microscopic units, large negative) when $p = 0$ and to the right (large positive) when $p = 1$. In both cases, $\alpha = 0$ by (9), hence by (13), $\overline{X}_t = \hat{x}_t(\omega)$ in the first case, and $\overline{X}_t = \hat{y}_t(\omega)$ in the second case. Therefore

$$\hat{x}_t(\omega) \gg 0 \gg \hat{y}_t(\omega). \quad (14)$$

We want to find idealized conditions under which a measurement protocol produces measurements that follow Born's rule exactly.

In thermodynamics, we get idealized relations in the thermodynamic limit of infinite size, which are still applicable with good accuracy to systems of small but macroscopic size. Similarly, in kinetic theory, the scattering matrix, defined through an asymptotic limit of times $t \rightarrow \pm\infty$ (and the associated infinite separability of clusters) is used to define with good accuracy the collision rates and products of microscopic scattering events (where distances are small but large compared to atomic distances and times are short but large compared to the time needed to travel an atomic distance) figuring in the derivation of the kinetic equations.

This justifies that we idealize, in the present situation, macroscopic distances and times as infinite and therefore assume in place of (14) the exact but idealized limit

$$\hat{x}_t(\omega) \rightarrow \infty, \quad \hat{y}_t(\omega) \rightarrow -\infty \quad \text{for } t \rightarrow \infty. \quad (15)$$

The desired idealized conditions for the emergence of Born's rule are now given by the following theorem.

Theorem. Let $\hat{x}_t, \hat{y}_t, \hat{z}_t$ be time-dependent random variables such that (15) holds and

$$\hat{v}_t(\omega) := \frac{\hat{z}_t(\omega)}{\hat{x}_t(\omega)} \rightarrow 0 \quad \text{for } t \rightarrow \infty. \quad (16)$$

and define the time-dependent random variable \hat{u}_t by

$$\hat{u}_t(\omega) := \frac{\hat{x}_t(\omega)}{\hat{x}_t(\omega) - \hat{y}_t(\omega)}.$$

If the limiting random variable

$$u := \lim_{t \rightarrow \infty} \hat{u}_t, \quad (17)$$

exists almost everywhere and is uniformly distributed in $[0, 1]$ then

$$\Pr(\bar{X}_t \rightarrow \infty) = p, \quad \Pr(\bar{X}_t \rightarrow -\infty) = 1 - p. \quad (18)$$

Indeed, under the stated conditions, $\hat{y}_t/\hat{x}_t = 1 - 1/\hat{u}_t$, hence

$$\bar{X}_t = \hat{x}_t \left(p + (1 - p)\hat{y}_t/\hat{x}_t + 2 \operatorname{Re} \alpha \hat{z}_t/\hat{x}_t \right) = x \left(1 - (1 - p)/\hat{u}_t + 2 \operatorname{Re} \alpha \hat{v}_t \right).$$

Thus

$$\Pr(\bar{X}_t \rightarrow \infty) = \Pr(1 - (1 - p)/u > 0) = \Pr(u > 1 - p) = p,$$

$$\Pr(\bar{X}_t \rightarrow -\infty) = \Pr(1 - (1 - p)/u < 0) = \Pr(u < 1 - p) = 1 - p.$$

Note that for approximately satisfying Born's rule it suffices that the assumptions are satisfied only approximately. Real detectors for microscopic events often magnify tiny initial displacements in a single scattering event (a single escaping electron in a photomultiplier or a single chemical reaction on a photographic plate) by special processes, thus making the infinite time limit irrelevant. Moreover, real detectors have various inefficiencies that may cause deviations from the ideal probabilistic law expressed by Born's rule.

Assumption (16) has the nature of a decoherence condition and is likely to be satisfied under quite general conditions, using a random phase approximation argument. The condition that u exists and is approximately uniformly distributed in $[0, 1]$ is the essential condition which requires a thorough analysis and must be verified in each concrete setting. It is likely that in the cases treated by AB&N and B&P discussed in Part III [13], such an analysis can be abstracted from their treatment.

4 Measurement errors

In this section, we give a detailed analysis of the concept of measurement error. This leads to a justification and comparison of the convention used to define measurement accuracy in the thermal interpretation with the traditional convention. It is followed by an analysis of the double slit experiment, which exemplifies the crucial differences of these conventions.

4.1 Defining measurement errors

Measurement errors are ubiquitous in physical practice; their definition requires, however, some care. A single measurement produces a number, the **measurement result**. The splitting of the measurement result into the sum of an intended result (the true value) and a **measurement error** (the deviation from it) depends on what one declares to be the true value. Thus what can be said about measurement errors depends on what one regards as the true value of something measured. This true value is a theoretical construct, an idealization arrived at by convention.

Since measured are only actual results, never the hypothesized true values, there is no way to determine experimentally which convention is the right one. Both the quantum formalism and the experimental record are independent of what one declares to be the true value of a measurement. Different conventions only define different ways of bookkeeping, i.e., different ways of splitting the same actual measurement results into a sum of true values and errors, in the communication about quantum predictions and experiments. Nothing in the bookkeeping changes the predictions and the level of their agreement with experiment.

Thus the convention specifying what to consider as true values is entirely a matter of choice, an **interpretation**. The convention one chooses determines what one ends up with, and each interpretation must be judged in terms of its implications for convenience and accuracy. Like conventions about defining measurement units [1], interpretations can be adjusted to improvements in theoretical and experimental understanding, in order to better serve the scientific community.

Born's statistical interpretation of quantum mechanics gives the following convention for the prediction of measurement results for measuring a quantity given by a self-adjoint operator A . One computes a number of possible idealized measurement values, the eigenvalues of A , of which one is exactly (according to most formulations) or approximately (if level spacings are below the measurement resolution) measured, with probabilities computed from A and the density operator ρ by the probability form of Born's rule. Thus the eigenvalues are the true values of Born's statistical interpretation.

Because of the critique of Born's rule given in Part I [11], the thermal interpretation explicitly rejects the part of Born's rule that declares the eigenvalues of operators as the true

values in a measurement. It differs from the tradition created in 1927 by Jordan, Dirac, and von Neumann, and proclaims in direct opposition the alternative convention that one computes a single possible idealized measurement value, the q-expectation

$$\bar{A} := \langle A \rangle := \text{Tr } \rho A$$

of A , which is approximately measured. Thus the true values of the thermal interpretation are the q-expectations rather than the eigenvalues.

Both interpretations are in full agreement with the experimental record: The same number obtained by a measurement may be interpreted in a dual way: It both measures some random eigenvalue to high (in the idealization even infinite) accuracy, and it simultaneously measures the q-expectation to low accuracy. In both cases, the measurement involves an additional uncertainty related to the degree of reproducibility of the measurement, given by the standard deviation of the results of repeated measurements. Tradition and the thermal interpretation agree in that this uncertainty is at least

$$\sigma_A := \sqrt{\langle A^2 \rangle - \langle A \rangle^2}.$$

This leads, among others, to Heisenberg's uncertainty relation.

4.2 What should be the true value?

As an illustration of the differences in the interpretation we first consider some piece of digital equipment with 3 digit display measuring some physical quantity X using N independent measurements. Suppose the measurement results were 6.57 in 20% of the cases and 6.58 in 80% of the cases. Every engineer or physicist would compute the mean $\bar{X} = 6.578$ and the standard deviation $\sigma_X = 0.004$ and conclude that the true value of the quantity X deviates from 6.578 by an error of the order of $0.004N^{-1/2}$.

Next we consider the measurement of a Hermitian quantity $X \in \mathbb{C}^{2 \times 2}$ of a 2-state quantum system in the pure up state, using N independent measurements, and suppose that we obtain exactly the same results. The thermal interpretation proceeds as before and draws the same conclusion. But Born's statistical interpretation proceeds differently and claims that there is no measurement error. Instead, each measurement result reveals one of the eigenvalues $x_1 = 6.57$ or $x_2 = 6.58$ in an unpredictable fashion with probabilities $p = 0.2$ and $1 - p = 0.8$, up to statistical errors of order $O(N^{-1/2})$. For $X = \begin{pmatrix} 6.578 & 0.004 \\ 0.004 & 6.572 \end{pmatrix}$, both interpretations of the results for the 2-state quantum system are consistent with theory. However, Born's statistical interpretation deviates radically from engineering practice, without any apparent necessity.

Finally we consider the energy measurement of an unknown system with discrete, unknown energy levels $E_1 < E_2 < \dots$, assumed to be simple eigenvalues of the Hamiltonian. We

also assume that the system is in a pure state $a_1|E_1\rangle + a_2|E_2\rangle$, where the kets denote the eigenstates of the Hamiltonian and $|a_1|^2 = p$, $|a_2|^2 = 1 - p$; for simplicity, higher levels than the lowest two are assumed to be absent. As a consequence, the q-expectation of the energy (represented by the Hamiltonian) can be exactly calculated, giving $\bar{E} = pE_1 + (1-p)E_2$. The uncertainty of the energy can be exactly calculated, too, giving $\sigma_E = \sqrt{p(1-p)}|E_1 - E_2|$.

Something analogous holds for the measurement of any quantity of an arbitrary 2-state system, such as a spin. According to the experimental record, the response of a good detector is **quantized**. Thus the measurement results E are concentrated at two spots of the detector, just as what one gets when measuring a classical diffusion process in a double-well potential (see, e.g., HONGLER & ZHENG [7]). Thus this distribution is bimodal with two sharp peaks, with details depending on the detection method used and its resolution.

In a frequently used idealization that ignores the limited efficiency of a detector, the distribution may even be assumed to be binomial, with measurement results that take only one of two values E'_1 and E'_2 corresponding to the modes of the bimodal distribution. This idealization eliminates in particular the effects responsible for a detector efficiency of $< 100\%$ in real experiments.

According to the thermal interpretation, each measurement result E is taken to be an approximation of the true value \bar{E} , with an error $|E - \bar{E}|$ of order at least σ_E . In the limit of arbitrarily many repetitions, the mean value of the approximations approaches \bar{A} and their standard deviation approaches σ_E . The bimodal distribution of the measurement results is explained by environment-induced randomness and environment-induced dissipation, as for a classical, environment-induced diffusion process in a double-well potential.

According to Born's statistical interpretation in the standard formulation,⁴ "the measured result will be one of the eigenvalues", each actual measurement result E is claimed to be one of the the exact (in general irrational) value E_1 or E_2 , and there is no measurement error.⁵ However, the measurement result is not reproducible: Multiple repetition of the measurement results in a random sequence of values E_1 and E_2 , with probabilities p and $1 - p$, respectively. In the limit of arbitrarily many repetitions, the mean value of this sequence approaches \bar{A} and the standard deviation approaches σ_E .

⁴This is the formulation appearing in WIKIPEDIA [19]. GRIFFITHS & SCHROETER [5, p.133] declare, "If you measure an observable [...] you are certain to get one of the eigenvalues". PERES [15, p.95] defines, "each one of these outcomes corresponds to one of the eigenvalues of A ; that eigenvalue is then said to be the result of a measurement of A ". Textbooks such as NIELSEN & CHUANG [14, p.84f] seem to avoid the issue by not referring to eigenvalues at all. But their declaration, "Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. [...] The index m refers to the measurement outcomes that may occur in the experiment. [...] the probability that result m occurs", with a formula that summed over all m gives the value 1, still assumes that the values m are exact results – otherwise each of several approximations to the same intended result would have to be represented by a different M_m , and their summation would not give 1.

⁵In an – apparently nowhere explicitly discussed – more liberal reading of the Born rule, some additional measurement error might be acceptable. But then Born's rule is no longer about measurement but about idealized measurements, whose observations are theoretical numbers, not actual results. Thus the liberal reading of Born's rule would be a purely theoretical construct, silent about actual measurement results.

If the energy levels are exactly known beforehand (or if the "energy" actually represents a component of a spin variable), one can calibrate the pointer scale to make $E'_1 = E_1$ and $E'_2 = E_2$. Then, as long as one ignores the idealization error, both interpretations become experimentally indistinguishable. However, as already pointed out in Part I [11], in the more realistic case where energy levels are only approximately known and must be inferred experimentally – the common situation in spectroscopy. The thermal interpretation, in agreement with the standard recipes for drawing inferences from inaccurate measurement results, still gives a correct account of the actual experimental situation, while Born's statistical interpretation paints an inadequate, idealized picture only.

4.3 The double slit experiment

Consider the quantum system consisting of the screen and an external classical electromagnetic field. This is a very good approximation to many experiments, in particular to those where the light is coherent. According to the standard interpretation, the analysis (given, e.g., in the quantum optics book by MANDEL & WOLF [8, Chapter 9]) of the response of the electrons in the screen to the field gives a Poisson process for the electron emission, at a rate proportional to the intensity of the incident field. This is consistent with what is observed when doing the experiment with coherent light. A local measurement of the parameters of the Poisson process therefore provides a measurement of the intensity of the field.

In this analysis, there is nothing probabilistic or discrete about the field; it is just a term in the Hamiltonian of the system. Thus, according to the standard interpretation, the probabilistic response is in this case solely due to the measurement apparatus – the screen, the only quantum system figuring in the analysis. At very low intensity, the electron emission pattern appears event by event, and the interference pattern emerges only gradually. Effectively, the screen exhibits what is called **shot noise**: it begins to stutter like a motor when fed with gas at an insufficient rate. The stuttering of the screen cannot be due to discrete eigenvalues of an operator representing the intensity – the only operator appearing in the analysis by Mandel and Wolf is an electron momentum operator coupling to a classical field.

The classical external field discussed so far is of course only an approximation to the quantum electromagnetic field, and was only used to show that the discrete response is due to the detector, and only triggered by the interaction with a field. A field mediating the interaction must be present with sufficient intensity to transmit the energy necessary for the detection events; these are for coherent quantum light independent and Poisson distributed even in a full quantum analysis (given by MANDEL & WOLF [8, Section 12.10]). In the case of noncoherent quantum light, only the quantitative details change.

The discrete result appears just because each screen electron makes a very inaccurate random binary measurement of the incident field intensity. Each single spot in the gradually

appearing interference pattern is measurable to high accuracy, but this is a high accuracy measurement of the screen only, not of the field (or its particle content). The low accuracies refer to accuracies of the implied field intensity – namely one unit at the responding position and zero units elsewhere, while the true intensity is low but nonzero everywhere where the high intensity interference pattern would show up.

Accepting Mandel and Wolf’s detector analysis, nothing depends on the deterministic nature of the thermal interpretation. But the latter explains (see Subsection 3.4) why neglecting the environment results in probabilistic features at all, and causes the electrons to exhibit a binary response – remaining bound or escaping to a macroscopic distance where the effect can be magnified by a photomultiplier.

4.4 Quantum buckets

In the thermal interpretation, one assumes that a stuttering effect similar to the one discussed in the preceding subsection, when measuring a low intensity classical electromagnetic field by a photosensitive surface, appears whenever one measures any classical or quantum field at very low intensity, whether a photon field or an electron field or a silver field or a water field is considered.

The stuttering effect mentioned in Subsection 4.3 may be illustrated as follows. We consider measuring the rate of classical water flow into a basin by the number of buckets (of a fixed size) per unit time needed to keep the water at a roughly fixed level of height. As long as there is enough flow the bucket is very busy and the flow is measured fairly accurately. But at very low rates it is enough to occasionally take out one bucket full of water and the bucket number is a poor approximation of the flow rate unless one takes very long times.

By the same principle, quantum detectors such as photocells and Geiger counters act as **quantum buckets**. The sole fact that one has counters already implies that, whatever they measure, the measurements are forced by construction to be integers. This limits the attainable resolution of what is measured as in the example of the 3-digit counter from Subsection 4.2. If used to measure continuous flow, the uncertainty is always at least $1/2$ in the units used for the counting.

5 Currents and particles

In this section we look at how situations traditionally treated in terms of currents may be viewed in the thermal interpretation in terms of fields or, more precisely, currents.

After considering the notion of currents in general, we look at how they may be used to visualize particle decays.

We then consider the Stern–Gerlach experiment, one of the standard textbook examples used in the context of introducing Born’s rule. Here a silver beam is split by a magnetic field into two beams. These beams are observed to produce two spots of silver deposit on a screen. With the thermal interpretation, we may interpret this experiment either on the level of quantum field theory in terms of currents or by considering individual silver atoms in the beam. The former is the fundamental level and is treated in Subsection 5.3. The latter is approximate but elementary and is treated in Subsection 5.4.

5.1 Currents

We consider the example for the measurement of a current with a galvanometer. From a quantum field theoretical point of view, an electric current consists (in the situation to be discussed here) of the motion of the electron field in a wire at room temperature. The thermal interpretation says that at any level of description, one has an electron field, and the theoretically exact current density is described by the distribution-valued beable (q-expectation)

$$J^\mu(x) = \text{Tr } \rho j^\mu(x)$$

determined by the current operator $j^\mu(x) = -e : \bar{\psi}(x)\gamma^\mu\psi(x) :$. Here the colons denote normal ordering and ρ is the density operator describing (in the Heisenberg picture) the exact state of the universe. Denoting by \hbar the **Boltzmann constant**, we define the **entropy operator** of the universe by $S := -\hbar \log \rho$, so that $\rho = e^{-S/\hbar}$.

At this level of description there is no approximation at all; the latter is introduced only when one replaces the exact S by a numerically tractable approximation. At or close to thermal equilibrium, it is well-established empirical knowledge that we have

$$S \approx (H + PV - \mu N)/T;$$

equality defines exact equilibrium. We can substitute this (or a more accurate nonequilibrium) approximation into the defining formula for $J(x)$ to compute a numerical approximation.

Ignoring reading uncertainties, a galvanometer measures an electric current of the form

$$I(t) = \int dz h_t(z) \cdot J(x + z) \tag{19}$$

flowing at time t through a cross section of the galvanometer. Here $h_t(z)$ is a smearing function that is negligible for z larger than the size of the current-sensitive part of the galvanometer. The precise h can be found by calibration.

The smearing is needed for mathematical reasons to turn the distribution-valued current into an observable vector, and for physical reasons since the galvanometer is insensitive to

very high spatial or temporal frequencies. This smearing has nothing to do with coarse-graining: It is also needed in already coarse-grained classical field theories. For example, in hydromechanics, the Navier-Stokes equations generally have only weak (distributional) solutions that make numerical sense only after smearing.

Thus the quantum situation is in the thermal interpretation not very different from the classical situation. In particular, nowhere was made use of any statistical argument; the trace (which in traditional statistical mechanics gets a statistical interpretation) is simply a calculational device for managing the q-expectations.

5.2 Particle decay

For other currents everything is analogous. It will be shown elsewhere that one can canonically associate to every bound state of a Poincaré invariant relativistic or Galilei invariant nonrelativistic quantum field theory a distinguished effective 4-vector current operator. This allows one to represent all asymptotic scattering phenomena at finite times using currents in place of particles.

In particular, in the thermal interpretation, currents provide the natural description for chemical reactions, collision processes, and particle decay, using the general picture justified in Subsections 4.2 and 5.1 of Part III [13] that discrete events emerge from coarse-graining through dissipation together with the discrete basin structure of the slow manifold of a physical system.

We explain the principle by considering a particle decay $A \rightarrow B + C$, such as $\pi^+ \rightarrow \mu^+ + \nu_\mu$. Note that at present, this only gives an intuitive picture of what should happen. The details of this thermal interpretation picture are still conjectural and need to be justified by future analysis of specific models.

At each time t one has three operator-valued effective 4-currents, one for each possibly flowing substance A, B, C . When the center of the reaction is at the origin, the reaction $A \rightarrow B + C$ proceeds as follows: At large negative times the A -density (q-expectation of the time component of the 4-current) is concentrated along the negative z -axis, and the A -current (q-expectation of the 3-vector of space components of the 4-current) is concentrated along the positive z -axis; the B -current and the C -current essentially vanish.

If the reaction happened (which depends on the details of the environment) then, at large positive times, the A -current is negligible, the B -density and C -density are concentrated along two (slightly diverging) rays emanating from the origin in such a way that momentum conservation holds, and the B -current and C -current are concentrated along these rays, too. Otherwise, at large positive times, the A -density is concentrated along the positive z -axis, and the A -current is concentrated along the positive z -axis, too, and the B -current and the C -current remain negligible. During the reaction time, i.e., when the fields are concentrated

near the origin, one can interpolate the asymptotic happening in an appropriate way. The details are defined by the interaction.

The manifold of slow modes splits into a basin corresponding to the decayed state (with two continuous angle parameters labeling the possible modes) one basin corresponding to the undecayed state. The metastable transition state at time zero determines together with the environmental fluctuations which basin is chosen and which direction is taken. This is comparable to what happens to bending a classical thin iron bar through longitudinal pressure in a random direction, though in that case the bar must bend, so that there is only one basin, with modes labelled by a single angle. In both cases, one of the continuous labels appears due to the rotational symmetry of the setting around the z-axis. In the case of the decay reaction, the second continuous label arises through another, infinitesimal symmetry at the saddle point at the origin.

This is one of the possible scenarios, probably what happens if the decay happens inside a dense medium (a secondary decay in a bubble chamber, say).

For a collision experiment in vacuum, there is probably not enough environmental interaction near zero, and after reaching the collision region, the *B*-current and the *C*-current should, in case a reaction happens, rather take a rotationally symmetric shape. In this case, the path like particle nature appears only later when the spherical fields reach a detector. The metastability of the detector forces the two spherical fields to concentrate along two paths, and momentum conservation makes these paths lie weighted-symmetric to the z-axis (would be geometrically symmetric when the decay products have equal mass). The details are essentially those reported in the 1929 paper by MOTT [10].

In both cases, the detection process creates the seeming particle nature of the observation record; cf. the discussion in Subsection 4.4 of Part I [11].

5.3 The Stern–Gerlach experiment in terms of currents

In the traditional analysis of the Stern–Gerlach experiment in terms of single silver particles, the dynamics is treated semiclassically for simplicity, and two beams appear as the only possible pathways.

In a field theoretic treatment, the beam is not interpreted classically but as a quantum field.⁶ Thus the silver is treated as an effective spinor field. It is not a free field because of the magnetic field in the experiment. The magnetic field is (in the usual semiclassical treatment) a term in the Hamiltonian of the field theory that changes the dynamics. It treats different components of the spinor field representing silver in opposite ways, turning a single beam at the source into two while passing the magnet.

⁶This is the difference to Schrödinger's failed early attempts to give a continuum interpretation of quantum mechanics in terms of classical fields.

Subsection 5.1 applies, except that the electric current is replaced by a silver current (which means that the formula defining it is a complicated multibody current). To get the total amount of silver deposited one also needs to integrate over the time of the experiment. Thus the effective support of the silver current operator $j(x)$ is initially along a single beam, which, upon entering the magnetic field, splits into two beams. The current flows along the direction of the two beams. The amount of silver on the screen at the end measures the integrated beam intensity, the total transported mass. This is in complete analogy to the qubit treated in Subsection 3.5 of Part III [13]. Particles need not be invoked.

The intensity of silver flow is the function of the position on the screen defined by the q -expectation of the incident current integrated over a spot centered at this position. Given the setup, the intensity is positive at the two spots predicted by the mathematics of the theory, and zero elsewhere.

The density operator is that of the whole universe, and the integration in (19) is effectively over a cell to which a piece of the equipment responds, done after the trace computation. The operation $\text{Tr } \rho j(x)$ yields a current J that is nonzero only at two small spots of any cross section (e.g., on the screen), and integrating over each spot gives in the symmetric case a total intensity of half of the original beam (before the apparatus) in each spot. Integrating over other regions of the screen gives zero since the integrand is zero there. This is why the silver flows into these two spots and nowhere else.

Thus when firing a continuous beam of high intensity one sees two spots, both appearing at essentially the same time. What is measured by a spot is the intensity of the silver flow into the spot, not the spin of single electrons.⁷

In the very low density case, the stuttering effect discussed in Subsection 4.3 for the double slit experiment becomes visible at a screen of sufficiently high resolution, and the response of the screen becomes erratic. In particular, if a beam contains only a single particle, the quantum field representing the beam is in a state with sharp particle number $N = 1$, but otherwise nothing changes. Conservation of mass, together with the instability of macroscopic superpositions and randomly broken symmetry forces that only one of the two spots gets marked by a silver atom, just as a classical bar under vertical pressure will bend into only one direction. It is not clear how Nature achieves the former, but this lack of explanation is common to all interpretations of quantum physics.

We may interpret the stuttering in terms of the quantum bucket picture from Subsection 4.4. We may think of each of the two spots on the screen as a quantum bucket counting impinging silver flow. We combine the counts into a single pointer variable x by counting left spot events downwards (-1) and right spot events upwards ($+1$). Each single atom

⁷The original Stern–Gerlach paper (and the early discussion about it) indeed talked about "Richtungsquantelung" (quantization of directions) and not of spin measurement. The fact that two beams appear is a consequence of the spin of the electron field, but has nothing per se to do with measuring an electronic spin state. The latter is defined only for single electrons, not for the electron field.

deposited somewhere on the screen is one bucket event reducing the intensity of the inflowing bilocal silver field. It approximates the true value in $[-1, 1]$ (the q-expectation of x) by either $+1$ or -1 , the only possible bucket results. This holds for every single atom, and hence for all the silver that arrives in the two spots. If we assume for simplicity that the silver source is prepared in a state where the q-expectation of x vanishes, taking the single buckets as measurement results each time in a binary measurement of the true (theoretically predicted) uncertain number 0 ± 1 , consistent with the measurement error of 1 in each case. This is completely independent of the flow rate.

5.4 The Stern–Gerlach experiment in terms of particles

An dem Tatbestand, die Elektronenschwärme betreffend, wie er bisher beschrieben wurde, ist nichts Paradoxes. Statt vom Schwarm spreche ich in Zukunft vom einzelnen Elektron und demgemäß von Wahrscheinlichkeit statt von Häufigkeit. Etwas Paradoxes liegt erst in der Aussage, daß σ_x die Komponente eines gewissen Vektors, des Impulsmomentes, in bezug auf die x -Richtung ist. Denn dies involviert doch, wenn wir ein rechtwinkliges Koordinatensystem $x y z$ im Raume einführen und die willkürliche Richtung r die Richtungskosinus a, b, c hat, die Gleichung

$$\sigma_r = a\sigma_x + b\sigma_y + c\sigma_z.$$

Wie verträgt sich das mit dem Umstand, daß σ_r so gut wie $\sigma_x, \sigma_y, \sigma_z$ nur der Werte ± 1 fähig ist?

Hermann Weyl, 1927 [18, p.8f]

We now consider the Stern–Gerlach experiment not in terms of a field measurement but as a spin measurement experiment of single silver atoms in the beam. In this case one must – like in every introductory text – treat the silver source as producing an ensemble of single atoms and, ignoring efficiency considerations, assume that each silver atom produces a tiny dot at one of the two spots on the screen.

The thermal interpretation looks at the reduced dynamics of the relevant macroscopic q-expectations and would find by a similar analysis as that in Section 3 that due to the reduced dynamics of the pointer variable (here the relative position of the condensed silver atom), all but the positions at the two spots are unstable, so that the total system is bistable.

Ignoring a factor of $\hbar/2$, we represent the spin measurement as a measurement of the q-expectations $\langle \sigma_3 \rangle \in [-1, 1]$ by means of a binary measurement of the spot on which an arriving silver atom is located, with possible values left spot (-1) or right spot ($+1$). In the thermal interpretation, each single dot on the screen, at either the left spot (-1) or the right spot ($+1$), is viewed as an approximate measurement of the q-expectation, which lies somewhere in $[-1, 1]$. This approximation is very poor. For example, when the initial state of the silver atoms is such that its q-expectations is $\langle \sigma_3 \rangle = 0$, the error of both binary measurement results ± 1 is 1, but with random signs, consistent with the computed uncertainty, which is also 1.

To improve the accuracy one needs to average over multiple measurement, and gets better results that converge to the true value 0 as the sample size gets arbitrarily large. To see this, one must consider a different operator, namely the mean spin $s = N^{-1}(s_1 + \dots + s_N)$, where s_k is the σ_3 of the k th silver atom in the ensemble measured. This mean spin operator has an associated (theoretically predicted) uncertain value of $\bar{s} \pm \sigma_s = 0 \pm N^{-1/2}$, which is approximately measured by the mean of the bucket results. This mean is for large N distributed as a Gaussian with zero mean and standard deviation $N^{-1/2}$, matching the prediction.

Born's statistical interpretation treats the measured position of each individual silver atom instead as an exact measurement of the discrete value ± 1 of the corresponding atom, with random signs. Although each single measurement is deemed error-free, the statistical uncertainty resulting from this randomness is still 1.

Clearly, both interpretations account for the same experimental facts, but in different ways. They make very different assumptions concerning the nature of what is to be regarded as the idealized measurement result to which the actual result is to be compared.

References

- [1] Bureau International des Poids et Mesure, The International System of Units (SI), 9th edition 2019, Draft, 18 December 2019.
<https://www.bipm.org/utils/en/pdf/si-revised-brochure/Draft-SI-Brochure-2018.pdf> [3, 19]
- [2] H.B. Callen. Thermodynamics and an introduction to thermostatistics, 2nd. ed., Wiley, New York, 1985. [6]
- [3] P.A.M. Dirac, The Principles of Quantum Mechanics, 1st ed., Oxford Univ. Press, Oxford 1930. [11]
- [4] A. Einstein, Elementare Überlegungen zur Interpretation der Grundlagen der Quanten-Mechanik, pp. 33–40 in: Scientific Papers Presented to Max Born, Oliver & Boyd, Edinburgh 1953. [7]
- [5] D.J. Griffiths and D.F. Schroeter, Introduction to Quantum Mechanics, Cambridge University Press, Cambridge 2018. [21]
- [6] P. Ehrenfest, Bemerkung über die angenäherte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik. Z. Physik A Hadrons and Nuclei, 45 (1927), 455–457. [8, 9]

- [7] M.O. Hongler and W.M. Zheng, Exact solution for the diffusion in bistable potentials, *J. Statist. Phys.*, 29 (1982), 317–327. [21]
- [8] L. Mandel and E. Wolf, *Optical coherence and quantum optics*, Cambridge Univ. Press, Cambridge 1995. [22]
- [9] K. Maurin, *General Eigenfunction Expansions and Unitary Representations of Topological Groups*, Polish Scientific Publishers, Warsaw 1968. [10]
- [10] N.F. Mott, The wave mechanics of α -ray tracks, *Proc. Royal Soc. London A* 126 (1929), 79–84. [26]
- [11] A. Neumaier, *Foundations of quantum physics I. A critique of the tradition*, Manuscript (2019). <https://arxiv.org/abs/1902.10778> [19, 22, 26]
- [12] A. Neumaier, *Foundations of quantum physics II. The thermal interpretation*, Manuscript (2019). <https://arxiv.org/abs/1902.10779> [3, 13]
- [13] A. Neumaier, *Foundations of quantum mechanics III. Measurement*, Manuscript (2019). <https://arxiv.org/abs/1902.10782> [3, 10, 18, 25, 27]
- [14] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*, Cambridge Univ. Press, Cambridge 2011. [21]
- [15] A. Peres, *Quantum theory: Concepts and methods*, Kluwer, Dordrecht 2002. [21]
- [16] M. Planck, Entropy and temperature of radiant heat, *Annalen der Physik* 1 (1900), 719–737. [11]
- [17] W. Ritz, *Recherches critiques sur l'électrodynamique générale*, *Annales Chemie et de Physique* 13 91908), 145–275. [11]
- [18] H. Weyl, *Quantenmechanik und Gruppentheorie*, *Z. Phys.* 46 (1927), 1–46. [28]
- [19] Wikipedia, Born rule, Web document (accessed April 24, 2019). https://en.wikipedia.org/wiki/Born_rule [21]