

Foundations of quantum physics

V. Coherent foundations

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Abstract. This paper is a programmatic article presenting an outline of a new view of the foundations of quantum mechanics and quantum field theory. In short, the proposed foundations are given by the following statements:

- Coherent quantum physics is physics in terms of a coherent space consisting of a line bundle over a classical phase space and an appropriate coherent product.
- The kinematical structure of quantum physics and the meaning of the fundamental quantum observables are given by the symmetries of this coherent space, their infinitesimal generators, and associated operators on the quantum space of the coherent space.
- The connection of quantum physics to experiment is given through the thermal interpretation. The dynamics of quantum physics is given (for isolated systems) by the Ehrenfest equations for q-expectations.

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1 Introduction

This paper, the fifth of a series of papers [42, 43, 44, 45] on the foundations of quantum physics, and the third one of a series of papers (NEUMAIER [39]) on coherent spaces and their applications, presents a new view of the foundations for quantum mechanics and quantum field theory, highlighting the problems and proposing solutions. In short, the proposed **coherent foundations** are given by the following statements, made precise later:

Coherent quantum physics is physics in terms of a coherent space consisting of a line bundle over a classical phase space and an appropriate coherent product. The kinematical structure of quantum physics and the meaning of the quantum observables¹ are given by the symmetries of this coherent space, their infinitesimal generators, and associated operators on the quantum space of the coherent space.

The connection of quantum physics to experiment is given through the thermal interpretation. The dynamics of quantum physics is given (for isolated systems) by the Ehrenfest equations for q-expectations.

The coherent foundations proposed here in a programmatic way resolve the problems with the traditional presentation of quantum mechanics discussed in Part I [42].

This paper is a programmatic overview article containing the main ideas on coherent spaces and their relation to quantum physics, not the precise concepts. These are defined and studied in depth in other papers of the series on coherent spaces, beginning with NEUMAIER [40] and NEUMAIER & GHAANI FARASHAHI [41]. See also the exposition at the web site NEUMAIER [39].

Section 2 gives rigorous definitions of the most basic concepts and results on coherent spaces, without attempting to be comprehensive, and (together with the next section) a general outline of a coherent quantum physics, telling the main points of the story with as few formulas and conceptual details as justifiable.

Section 3 introduces the concept of symmetries (invertible coherent maps) of coherent spaces and associated quantization procedures. This leads to quantum dynamics, which in special (completely integrable) situations can be solved in closed form in terms of classical motions on the underlying coherent space, if the latter has a compatible manifold structure. Spectral issues can in favorable cases be handled in terms of dynamical Lie algebras. Close relations to concepts from geometric quantization and Kähler manifolds are pointed out.

In Section 4, we rephrase the formal essentials of the thermal interpretation in a slightly generalized more abstract setting, to emphasize the essential mathematical features and

¹In the following, these will be called **quantities or q-observables to distinguish them from observables in the operational sense of numbers obtainable from observation. Similarly, we use at places q-expectation for the expectation value of quantities.**

the close analogy between classical and quantum physics. We show how the coherent variational principle (the Dirac–Frenkel procedure applied to coherent states) can be used to show that in coarse-grained approximations that only track a number of relevant variables, quantum mechanics exhibits chaotic behavior that, according to the thermal interpretation, is responsible for the probabilistic features of quantum mechanics.

The final Section 5 defines the meaning of the notion of a field in the abstract setting of Section 4 and shows how coherent spaces may be used to define relativistic quantum field theories.

The puzzle of making sense of the foundations of quantum physics held my attention for many years. Around 2003, I discovered that group coherent states are for many purposes very useful objects; before, they were for me just a facet that physicists studied who needed them for quantum optics. In 2007, I realized that apparently all of quantum mechanics and quantum field theory can be profitably cast into this form, and that coherent states may provide better theoretical foundations for quantum mechanics and quantum field theory than the current Fock space approach. Since then I have been putting piece by piece into the new framework, and always found (after some work) everything nicely fitting. With each new piece in place, I got insights about how to interpret everything, and things got simpler and simpler as I proceeded. Or rather, more and more complicated things became understandable without significant increase of complexity in the new picture. Everything became much more transparent and intuitive than the traditional mental picture of quantum physics was.

In the bibliography, the number(s) after each reference give the page number(s) where it is cited.

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2 Coherent spaces

Coherent quantum physics is quantum physics in terms of a coherent space consisting of a classical phase space and an appropriate coherent product. The kinematical structure and the meaning of the quantities are given by the symmetries (invertible coherent maps) of the coherent space.

This section gives rigorous definitions of the most basic concepts and results, without attempting to be comprehensive, and (together with the next section) a general outline of a coherent quantum physics, telling the main points of the story with as few formulas and conceptual details as justifiable. Unexplained details can be found in my papers on

coherent spaces (and the references given there). Two of these papers (NEUMAIER [40] and NEUMAIER & GHAANI FARASHAHI [41]) are already publicly available; others are in preparation and will become available at my web site [39].

Coherent spaces are a novel mathematical concept, a nonlinear version of Hilbert spaces. They combine the rich, often highly characteristic variety of symmetries of traditional geometric structures with the computational tractability of traditional tools from numerical analysis and statistics.

To get the axioms of a coherent space from those of a Hilbert space, the vector space axioms are dropped while the notion of inner product and its properties is kept. Every subset of a real or complex Hilbert space may be viewed as a coherent space. Symmetries induced by orthogonal resp. unitary transformations become symmetries of the coherent space.

Conversely, every coherent space can be canonically embedded into a complex Hilbert space (namely its quantum space) in such a way that all its symmetries are realized by unitary transformations. Thus, in a way, the theory of coherent spaces is just the theory of subsets of a Hilbert space and their symmetries. However, just as it pays to study the properties of manifolds independently of their embedding into a Euclidean space, so it appears fruitful to study the properties of coherent spaces independent of their embedding into a Hilbert space.

There are close connections to reproducing kernel Hilbert spaces, leading to numerous applications in quantum physics, complex analysis, statistics, and stochastic processes.

One of the strengths of the coherent space approach is that it makes many different things look alike, and stays close to actual computations. There are so many applications in physics and elsewhere that pointing them all out will take a whole book to write. . .

Coherent states and squeezed states in quantum optics, mean field calculations in statistical mechanics, Hartree–Fock calculations for the electronic states of atoms, semiclassical limits, integrable systems all belong here. As will be shown in later papers from this series, most computational techniques in quantum physics can be profitably phrased in terms of coherent spaces.

2.1 Coherent spaces

Fundamental is the notion of a coherent space. It is a nonlinear version of the notion of a complex Hilbert space: The vector space axioms are dropped while the notion of inner product, now called a coherent product, is kept. Every coherent space can be embedded into a Hilbert space extending the coherent product to an inner product.

In informal, traditional terms, a coherent space is roughly a set Z whose elements label

certain vectors, called **coherent states** of a Hilbert space. The quantum space of Z is the closed subspace formed by the limits of linear combinations of coherent states.

However, one can characterize this situation independent of a Hilbert space setting. Then a coherent space is a set Z equipped (among others) with a so-called coherent product that assigns to any two points $z, z' \in Z$ a complex number $K(z, z')$ satisfying certain coherence properties. The coherent product is essentially the inner product in the quantum space of the coherent states with the corresponding classical labels.

More formally, a **Euclidean space** is a complex vector space \mathbb{H} with a binary operation that assigns to $\phi, \psi \in \mathbb{H}$ the **Hermitian inner product** $\phi^*\psi \in \mathbb{C}$, antilinear in the first and linear in the second argument, such that

$$\overline{\phi^*\psi} = \psi^*\phi, \quad (1)$$

$$\psi^*\psi > 0 \quad \text{for all } \psi \in \mathbb{H} \setminus \{0\}. \quad (2)$$

In physics, one usually writes $\langle \phi | \psi \rangle$ in place of $\phi^*\psi$, but we reserve this bra-ket notation exclusively for coherent states, as defined below. \mathbb{H} has a natural locally convex topology in which the inner product and any linear functional is continuous, and is naturally embedded into its antidual \mathbb{H}^\times , the space of antilinear functionals on \mathbb{H} . The Hilbert space completion $\overline{\mathbb{H}}$ sits between these two spaces,

$$\mathbb{H} \subseteq \overline{\mathbb{H}} \subseteq \mathbb{H}^\times.$$

$\text{Lin}^\times \mathbb{H}$ denotes the space of linear mappings from \mathbb{H} to \mathbb{H}^\times ; they are automatically continuous.

A **coherent space** is a nonempty set Z with a distinguished function $K : Z \times Z \rightarrow \mathbb{C}$, called the **coherent product**, such that

$$\overline{K(z, z')} = K(z', z), \quad (3)$$

and for all $z_1, \dots, z_n \in Z$, the $n \times n$ matrix G with entries $G_{jk} = K(z_j, z_k)$ is positive semidefinite.

The **distance** (PARTHASARATHY & SCHMIDT [48])

$$d(z, z') := \sqrt{K(z, z) + K(z', z') - 2 \text{Re} K(z, z')} \quad (4)$$

of two points $z, z' \in Z$ is nonnegative and satisfies the triangle inequality. The distance is a metric precisely when the coherent space is **nondegenerate**, i.e., iff

$$K(z'', z') = K(z, z') \quad \forall z' \in Z \quad \Rightarrow \quad z'' = z.$$

In the resulting topology, the coherent product is continuous.

A **coherent manifold** is a smooth ($= C^\infty$) real manifold Z with a smooth coherent product $K : Z \times Z \rightarrow \mathbb{C}$ with which Z is a coherent space. In a nondegenerate coherent manifold, the infinitesimal distance equips the manifold with a canonical Riemannian metric.

A **quantum space** $\mathbb{Q}(Z)$ of Z is a Euclidean space spanned (algebraically) by a distinguished set of vectors $|z\rangle$ ($z \in Z$) called **coherent states** satisfying

$$\langle z|z'\rangle = K(z, z') \quad \text{for } z, z' \in Z \quad (5)$$

with the linear functionals

$$\langle z| := |z\rangle^*$$

acting on $\mathbb{Q}(Z)$. Coherent states with distinct labels are distinct iff Z is nondegenerate.

A construction of ARONSZAJN [2, 3] (attributed by him to MOORE [36]), usually phrased in terms of reproducing kernel Hilbert spaces, proves the following basic result.

Moore–Aronszajn Theorem. *Every coherent space has a quantum space. It is unique up to isometry.*

The antidual $\mathbb{Q}^\times(Z) := \mathbb{Q}(Z)^\times$ of the quantum space $\mathbb{Q}(Z)$ is called the **augmented quantum space**. It contains the **completed quantum space** $\overline{\mathbb{Q}}(Z)$, the Hilbert space completion of $\mathbb{Q}(Z)$,

$$\mathbb{Q}(Z) \subseteq \overline{\mathbb{Q}}(Z) \subseteq \mathbb{Q}^\times(Z).$$

In quantum mechanical applications, $\overline{\mathbb{Q}}(Z)$ is the Hilbert space containing the pure states, while $\mathbb{Q}^\times(Z)$ also contains unnormalizable wave functions.

Constructing Hilbert spaces from a coherent space and its coherent product is much more flexible, and hence more powerful, than the standard approach of constructing Hilbert spaces from a function space and a measure on it. Virtually every Hilbert space arising in quantum mechanical practice can be neatly constructed as the quantum space of an appropriate coherent space; the preceding examples gave the first bits of evidence of this.

In a quantum mechanical context, Z is a classical phase space or extended phase space – typically a symplectic manifold, a Poisson manifold, or a circle or line bundle over such a manifold that incorporates the classical action variable (encoding the Berry phase under quantization). For example, the Aharonov–Bohm effect [1] needs the bundle formulation. A canonical symplectic form is determined by the coherent product. The precise relationship is the subject of **geometric quantization**, loosely outlined in Subsection 3.4.

This provides a classical view of the system. On the other hand, the coherent product also determines its quantum space, whose completion $\overline{\mathbb{Q}}(Z)$ is the Hilbert space of quantum mechanical state vectors. This provides a quantum view of the system.

Thus coherent spaces allow both a classical and a quantum view of the same system. The two views are closely related, as the phase space points $z \in Z$ label a family of **coherent**

states $|z\rangle$, special vectors in the quantum space for which the inner product takes the simple form

$$\langle z|z'\rangle = K(z, z'). \quad (6)$$

Thus in some sense, the classical phase space and the quantum Hilbert space coexist in the framework of coherent spaces. The classical phase space is a quotient space of Z under the equivalence relation that identifies points whose corresponding coherent states differ only by a scale factor. Thus points in the phase space are in 1-1 correspondence with equivalence classes of points of Z , hence equivalence classes of labels of coherent states. The quantum space is the completion of the space spanned by all coherent states. It is a Hilbert space that can be realized as a space of functions on Z ; the coherent states $|z\rangle$ are essentially the functions that map $z' \in Z$ to the coherent product $K(z, z')$.

If we regard Z as a classical phase space, as often adequate, the functions

$$\widehat{\psi}(z) := \psi^T |z\rangle, \quad \psi \in \mathbb{Q}(Z).$$

are those classical phase space functions that have an immediate quantum meaning. Note that $\mathbb{Q}^\times(Z)$ consists of all complex-valued maps on Z that are continuous in the natural weak topology induced by the coherent product.

Glauber coherent states (mentioned before) are a particular instantiation of this concept. A more trivial case to keep in mind is to label *all* vectors in finite-dimensional Hilbert space \mathbb{C}^n , so that $Z = \mathbb{C}^n$ and $\langle z|z'\rangle = K(\bar{z}, z')$ with

$$K(z, z') := z^* z' = \sum_k \bar{z}_k z'_k. \quad (7)$$

This extends to infinite dimensions (the usual case in most of quantum physics) by replacing the sum by an appropriate integral, and shows that the traditional way of looking at Hilbert spaces can be fully accommodated with such a coherent space. However, this choice is poor from the point of view of the classical-quantum correspondence. As we shall see, there are far better choices, leading to a much increased flexibility compared to the traditional approach of defining Hilbert spaces by giving the inner product as a sum or integral. More importantly, as one works most of the time in Z and very little explicitly in the quantum space, one can often use classical intuition in quantum situations, and the economy of classical computations is often preserved.

Finite linear combinations of coherent states form a dense subspace $\mathbb{Q}(Z)$ of the Hilbert space $\overline{\mathbb{Q}}(Z)$. This implies that all quantum mechanical calculations, usually done in an orthonormal basis, can also be done on the basis of coherent states, and often far more efficiently. Most conceptual issues can be discussed in coherent terms, too. This makes the closeness to a classical description very plain, and removes most of the mystery of quantum physics.

The simplest classical systems have a finite number N of states, corresponding to a phase space Z with N elements. Their dynamics is that of a hopping process, a **continuous time Markov chain** determined by consistently specifying transition rates for hopping from one state to another. More complex classical systems have phase spaces Z that are finite-dimensional manifolds when there are only finitely many degrees of freedom. In particular, this is the arena of **classical mechanics** of point particles, where Z is a symplectic manifold, or more generally a Poisson manifold. The deterministic dynamics is defined on Z by Hamilton's equations, equivalently on phase space functions by means of the Poisson bracket. Finally, in **classical field theory**, the phase space Z is an infinite-dimensional space of fields in 3-dimensional space, the deterministic dynamics on Z is described by partial differential equations. Often an equivalent dynamics on phase space functions (now functions on fields) is given in terms of an appropriate Poisson structure on Z .

The simplest quantum systems have a finite number N of levels, corresponding to a Hilbert space of dimension N . We may consider them as the quantum version of a Markov chain; this corresponds to picking an orthonormal basis of N pointer vectors $|z\rangle$ and declaring the coherent product to be $K(z, z') := \langle z|z'\rangle = \delta_{zz'}$, thus creating a coherent space Z with N elements. However, a 2-level quantum system also models a **spinning electron** at rest in its ground state. Here the appropriate classical analogue is not the counterintuitive two state (up-down) model which depends on a distinguished direction and hence sacrifices the spherical symmetry of the electron, but a 2-sphere in \mathbb{R}^3 , the phase space of a classical spinning top. To account for the nonintegral spin of the electron, we should in fact take as classical phase space the double cover of the 2-sphere, given by the unit sphere

$$Z = \{z \in \mathbb{C}^2 \mid z^*z = 1\}$$

in \mathbb{C}^2 . (The double cover is the so-called Hopf fibration, a nontrivial object.) The 3-sphere is the same thing as the unit sphere in \mathbb{C}^2 written in real coordinates. The discussion of the Hopf fibration in terms of quaternions can be interpreted in terms of Pauli matrices, giving the traditional approach to 2-level systems. In terms of coherent states, all these technicalities are hidden – one has the quantum space without having to bother about the latter. This economy of coherent states becomes more pronounced in more complicated models, which is the most important one of the reasons why they are studied here.

To get the correct 2-state quantum space, we need to take the trivial coherent product (7) restricted to Z . Remarkably, the case of a particle of **higher spin** j has the same phase space, with the coherent product only slightly changed to

$$K(z, z') := (z^T z')^{2j+1}. \tag{8}$$

Equally remarkably, the coherence conditions are satisfied for this coherent product only if $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, thus naturally accounting for the fact that spin is quantized.

In contrast, accounting for arbitrary spin in the traditional fashion based on a $(2j + 1)$ -level system requires a significant amount of machinery already to define the representation.

2.2 Examples

Example: Klauder spaces. The Klauder space $KL[V]$ over the Euclidean space V is the coherent manifold $Z = \mathbb{C} \times V$ of pairs $z := [z_0, \mathbf{z}] \in \mathbb{C} \times V$ with coherent product $K(z, z') := e^{\bar{z}_0 + z'_0 + \mathbf{z}^* \mathbf{z}'}$. ($KL[\mathbb{C}]$ is essentially in KLAUDER [30]. Its coherent states are precisely the nonzero multiples of those discovered by SCHRÖDINGER [52].) As shown in detail in NEUMAIER & GHAANI FARASHAHI [41], where coherent construction of creation annihilation operators together with their properties are derived, the quantum spaces of Klauder spaces are essentially the **Fock spaces** introduced by FOCK [16] in the context of **quantum field theory**. They were first presented by SEGAL [53] in a form equivalent to the above. The quantum space of $KL[\mathbb{C}^n]$ was systematically studied by BARGMANN [4].

Example: The Bloch sphere. The unit sphere in \mathbb{C}^2 is a coherent manifold Z_{2j+1} with coherent product $K(z, z') := (z^* z')^{2j}$ for some $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. It corresponds to the Poincaré sphere (or Bloch sphere) representing a single quantum mode of an atom with spin j , or for $j = 1$ the polarization of a single photon mode. The corresponding quantum space has dimension $2j + 1$. The associated coherent states are the so-called **spin coherent states**.

This example shows that the same set Z may carry many interesting coherent products, resulting in different coherent spaces with nonisomorphic quantum spaces.

Example: The classical limit. In the limit $j \rightarrow \infty$, the unit sphere turns into the coherent space of a classical spin, with coherent product

$$K(z, z') := \begin{cases} 1 & \text{if } z' = \bar{z}, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting quantum space is infinite-dimensional and describes **classical stochastic motion** on the Bloch sphere in the Koopman representation.

More generally, any coherent space Z gives rise to an infinite family of coherent spaces Z_n on the same set Z but with modified coherent product $K_n(z, z') := K(z, z')^n$ with a nonnegative integer n . (The need for a nonnegative integer is related to Bohr–Sommerfeld quantization.) The quantum space $\mathbb{Q}(Z_n)$ is the symmetric tensor product of n copies of the quantum space $\mathbb{Q}(Z)$. If Z has a physical interpretation and the classical limit $n \rightarrow \infty$ exists, it usually has a physical meaning, too.

The same abstract quantum system may allow different classical views. The most conspicuous expression of this ambiguity is the **particle-wave duality**, a notion describing the seemingly paradoxical situation that the same quantum system may be approximately interpreted either in terms of classical particles or in terms of classical waves, though depending on the circumstances only one of the approximate views may be accurate enough to be useful. This is accommodated by writing the same Hilbert space in different but isomorphic ways as the quantum space of different coherent spaces.

Complex quantum systems with finitely many degrees of freedom can be modeled on the same phase spaces as the corresponding classical systems, and with little additional conceptual effort. (Traditionally, one would need the second quantization formalism or a first quantized equivalent.) The possibility to describe **motion** is added by augmenting the state space by variables for position and momentum. Several particles are accounted for by taking the direct product of the single-particle phase spaces, the coherent products simply multiply.

Example: Subsets of a Euclidean space. Any subset Z of a Euclidean space \mathbb{H} is a coherent space with coherent product $K(z, z') := z^* z'$. If the linear combinations of Z are dense in \mathbb{H} , then $\overline{\mathbb{Q}}(Z) = \overline{\mathbb{H}}$. Conversely, any coherent space arises in this way from its quantum space.

Example: Quantum spaces of entire functions. A **de Branges function** (DE BRANGES [10]) is an entire analytic function $E : \mathbb{C} \rightarrow \mathbb{C}$ satisfying

$$|E(\bar{z})| < |E(z)| \quad \text{if } \text{Im } z > 0.$$

With the coherent product

$$K(z, z') := \begin{cases} \overline{E'}(\bar{z})E(z') - E'(\bar{z})\overline{E}(z') & \text{if } z' = \bar{z}, \\ \frac{\overline{E}(\bar{z})E(z') - E(\bar{z})\overline{E}(z')}{2i(\bar{z} - z')} & \text{otherwise,} \end{cases}$$

where $E'(z)$ denotes the derivative of $E(z)$ with respect to z , $Z = \mathbb{C}$ is a coherent space. The corresponding quantum spaces are the de Brange spaces relevant in complex analysis.

Coherent spaces and reproducing kernel Hilbert spaces are mathematically almost equivalent concepts, and there is a vast literature related to the latter. Most relevant for the present work are the books by PERELOMOV [50] and NEEB [37]; for applications in probability and statistics, see also BERLINET & THOMAS-AGNAN [8].

However, the emphasis in these books is quite different from the present exposition, as they are primarily interested in properties of the associated functions, while we are primarily interested in the geometry and symmetry properties and in computational tractability.

2.3 New states from old ones

From the set of coherent states it is possible to create a large number of other states whose inner product is computable by a closed formula. This is important for numerical applications, since one can pick from the new states created in this fashion a suitable subset and declare the states belonging to this subset to be the coherent states of a new, derived coherent space. This way of constructing new coherent spaces from old ones allows one to apply the general body of techniques for the analysis of coherent spaces and their quantum properties to the new coherent space. Many known numerical techniques for quantum physics problems become in this way organized in the same setting.

The first, often useful construction takes a path $u(t)$ in Z and creates new states

$$[R_t u(t)] := \lim_{h \downarrow 0} h^{-1} \left(|u(t+h)\rangle - |u(t)\rangle \right).$$

We write $= \partial_j K$ for the partial derivative with respect to the j th argument of K , and find the inner products

$$\begin{aligned} \langle z | [R_t u(t)] &= \lim_{h \downarrow 0} h^{-1} \left(K(z, u(t+h)) - K(z, u(t)) \right) = \partial_2 K(z, u(t)) \dot{u}(t), \\ [R_t u(t)]^* [R_s v(s)] &= \lim_{h \downarrow 0} h^{-1} \left(\langle u(t+h) | [R_s v(s)] - \langle u(t) | [R_s v(s)] \right) \\ &= \lim_{h \downarrow 0} h^{-1} \left(\partial_2 K(u(t+h), v(s)) \overline{\dot{v}(t)} - \partial_2 K(u(t), v(s)) \overline{\dot{v}(t)} \right) \\ &= \dot{u}(t) \partial_1 \partial_2 K(u(t), v(s)) \dot{v}(s). \end{aligned}$$

Similar expressions can be found by taking other smooth parameterizations of submanifolds of Z , and taking limits corresponding to first order or higher order partial derivatives.

A trivial construction is to take linear combinations

$$[\alpha, y] := \sum_k \alpha_k |y_k\rangle,$$

where α is a finite sequence of complex numbers α_k and y is a finite sequence of points $y_k \in Z$. The inner products are given by

$$\begin{aligned} \langle z | [\alpha, y] &= \sum_k \alpha_k K(z, y_k), \\ [\alpha, y]^* [\alpha', y'] &= \sum_{j,k} \alpha_j \alpha'_k K(y_j, y_k), \end{aligned}$$

This also works for infinite sequences provided the right hand sides are always absolutely convergent, and with sums replaced by integrals for weighted integrals $\int \alpha(x) |y(x)\rangle d\mu(x)$,

provided the corresponding integrals on the right hand sides are always absolutely convergent. Of course, all these recipes can also be combined.

We see that, unlike in traditional Hilbert spaces, where the calculation of inner products always requires to evaluate often high-dimensional integrals, here the calculation of inner products is much simpler, often only taking sums and derivatives.

3 Coherent spaces and quantization

This section introduces the concept of symmetries (invertible coherent maps) of coherent spaces and associated quantization procedures. This leads to quantum dynamics, which in special (completely integrable) situations can be solved in closed form in terms of classical motions on the underlying coherent space, if the latter has a compatible manifold structure. Spectral issues can in favorable cases be handled in terms of dynamical Lie algebras. Close relations to concepts from geometric quantization and Kähler manifolds are pointed out.

3.1 Symmetries

Symmetries of a coherent space are transformations of the space that preserve the coherent structure. They generalize canonical transformations of a symplectic manifold, which is the special case of classical mechanics of point particles. More specifically, a **symmetry** of a coherent space Z is a bijection A of Z with the property that

$$K(z, Az') = K(A^T z, z') \quad (9)$$

for another bijection A^T .

Let Z be a coherent space. A map $A : Z \rightarrow Z$ is called **coherent** if there is an **adjoint map** $A^* : Z \rightarrow Z$ such that

$$K(z, Az') = K(A^* z, z') \quad \text{for } z, z' \in Z \quad (10)$$

If Z is nondegenerate, the adjoint is unique, but not in general.

A **symmetry** of Z is an invertible coherent map on Z with an invertible adjoint.

Coherent maps form a semigroup $\text{Coh } Z$ with identity; the symmetries form a group.

An **isometry** is a coherent map A that has an adjoint satisfying $A^* A = 1$. An invertible isometry is called **unitary**.

Symmetries of a coherent space often represent the dynamical symmetries (see, e.g., BARUT & RACZKA [7]) of an associated exactly solvable classical system. For example, if Z is a

line bundle over a symplectic phase space, the symmetries would be all linear symplectic maps and their central extensions. (But only some of them preserve the Hamiltonian and hence are symmetries of the system with this Hamiltonian.)

In the coherent space formed by a subset Z of \mathbb{C}^n closed under conjugation, with coherent product $K(z, z') := z^T z'$, all $n \times n$ matrices mapping Z into itself are (in this particular case linear) coherent maps, and all invertible matrices are symmetries.

Example: Distance regular graphs. The orbits of groups of linear self-mappings of a Euclidean space define coherent spaces with predefined transitive symmetry groups. For example, the symmetric group $\text{Sym}(5)$ acts as a group of Euclidean isometries on the 12 points of the icosahedron in \mathbb{R}^3 . The coherent space consisting of these 12 points with the induced coherent product therefore has $\text{Sym}(5)$ as a group of unitary symmetries. The quantum space is \mathbb{C}^3 . The skeleton of the icosahedron is a distance-regular graph, here a double cover of the complete graph on six vertices. Many more interesting examples of finite coherent spaces are related to Euclidean representations of distance regular graphs and other highly symmetric combinatorial objects. See, e.g., BROUWER et al. [9].

The importance of coherent maps stems from the fact that there is a **quantization operator** Γ that associates with every coherent map A a linear operator $\Gamma(A)$ on the quantum space $\mathbb{Q}(Z)$. In the literature, when applied to the special case where $\mathbb{Q}(Z)$ is a Fock space, $\Gamma(A)$ is called the **second quantization** of A .

Quantization Theorem. *Let Z be a coherent space and $\mathbb{Q}(Z)$ a quantum space of Z . Then for any coherent map A on Z , there is a unique linear map $\Gamma(A) : \mathbb{Q}(Z) \rightarrow \mathbb{Q}(Z)$ such that*

$$\Gamma(A)|z\rangle = |Az\rangle \quad \text{for all } z \in Z. \quad (11)$$

*We call $\Gamma(A)$ the **quantization** of A and Γ the **quantization map**.*

The quantization map furnishes a representation of the semigroup of coherent maps on Z (and hence of the symmetry group) on the quantum space of Z . In particular, this gives a **unitary representation** of the group of unitary coherent maps on Z .

The quantization operator is important as it reduces many computations with coherent operators in the quantum space of Z to computations in the coherent space Z itself. By the quantization theorem, large semigroups of coherent maps produce large semigroups of coherent operators, which may make complex calculations much more tractable. Coherent spaces with many coherent maps are often associated with symmetric spaces in the sense of differential geometry.

This essentially means that symmetries are those invertible linear transformations of the quantum space that map coherent states into coherent states, but is expressed without

reference to the quantum space. This has very important implications for practical computations, reducing computations in the quantum space to simple computations in the coherent space. In particular, this makes certain problems easily exactly solvable that are in the traditional position or momentum representations nearly intractable. For example, the calculation of q-expectations requires in the traditional setting the evaluation of an integral over configuration space. In case of field theory, the configuration space is infinite-dimensional, and already a rigorous definition of such integrals is very difficult. Moreover, finding closed formulas for integrals in high or infinite dimensions is more an art than a science. In contrast, in the coherent space approach, many q-expectations of interest can be obtained by differentiation, which is a fully algorithmic process.

In case of the trivial coherent product (7), equation (9) holds for every $n \times n$ matrix with the usual matrix transpose. This motivates the general case, and shows in particular that the trivial coherent space has the **general linear group** $GL(n, \mathbb{C})$ of invertible complex $n \times n$ matrices as its group of symmetries. For virtually all quantum systems of interest there is a large classical **dynamical symmetry group**, which describes the symmetries of the underlying coherent space. Typically, this symmetry group is a (possibly infinite-dimensional) Lie group, much larger than the symmetry group of the system itself – which is the subgroup commuting with the Hamiltonian (in the nonrelativistic case) or preserving the action (in the relativistic case).

Example: Möbius space. The **Möbius space** $Z = \{z \in \mathbb{C}^2 \mid |z_1| > |z_2|\}$ is a coherent manifold with coherent product $K(z, z') := (\bar{z}_1 z'_1 - \bar{z}_2 z'_2)^{-1}$. A quantum space is the Hardy space of analytic functions on the complex upper half plane with Lebesgue integrable limit on the real line. The Möbius space has a large semigroup of coherent maps (a semigroup of compressions, OLSHANSKI [46]) consisting of the matrices $A \in \mathbb{C}^{2 \times 2}$ such that

$$\alpha := |A_{11}|^2 - |A_{21}|^2, \quad \beta := \bar{A}_{11}A_{12} - \bar{A}_{21}A_{22}, \quad \gamma := |A_{22}|^2 - |A_{12}|^2$$

satisfy the inequalities

$$\alpha > 0, \quad |\beta| \leq \alpha, \quad \gamma \leq \alpha - 2|\beta|.$$

It contains as a group of symmetries the group $GU(1, 1)$ of matrices preserving the Hermitian form $|z_1|^2 - |z_2|^2$ up to a positive factor.

Highest weight representations. The example of the Möbius space generalizes to a large class of exactly solvable classical systems with finitely many degrees of freedom, corresponding to the coherent states from group representations discussed in ZHANG et al. [59] and SIMON [54], which are close to being computable (though not all needed details are in these papers). The constructions relate to central extensions of all **semisimple Lie groups** and associated **symmetric spaces** or **symmetric cones** and their **line bundles**. These provide many interesting examples of coherent manifolds. This follows from work on

coherent states constructed from highest weight representations, discussed in monographs by PERELOMOV [50], by FARAUT & KORÁNYI [15], by NEEB [37].

Coherent states from highest weight representations induce on the corresponding coadjoint orbit a measure, a metric, a symplectic form, and an associated symplectic Poisson bracket. (See the ZHANG et al. [59] survey for details from a physical point of view. The Poisson bracket defines a Lie algebra on phase space functions (C^∞ functions on the coadjoint orbit, hence an associated group of Hamiltonian diffeomorphisms, and the coherent state approach effectively quantizes this group. All this can be reconstructed directly from the associated coherent spaces. In particular, the nonclassical states of light in quantum optics called **squeezed states** are described by coherent spaces corresponding to the **metaplectic group**; cf. related work by NERETIN [38].

3.2 q-observables and dynamics

We now assume that Z is a **coherent manifold**. This means that Z carries a C^∞ -manifold structure with respect to which the coherent product is smooth (C^∞). The relevant **observables** of the classical system are the discrete symmetries and the infinitesimal generators of the 1-parameter groups of symmetries that are smooth on the coherent product. They are promoted to q-observables of the corresponding quantum system through the quantization map. For a symmetry A , the corresponding q-observable is $\Gamma(A)$. For an **infinitesimal symmetry** X , i.e., an element of the Lie algebra of generators of 1-parameter groups of the symmetry group), the corresponding **quantum symmetry**, acting on the quantum space of Z , is the q-observable given by the strong limit

$$d\Gamma(X) := \lim_{s \downarrow 0} \frac{\Gamma(e^{isX}) - 1}{is}.$$

Note that

$$d\Gamma(X + Y) = d\Gamma(X) + d\Gamma(Y), \quad e^{d\Gamma(X)} = \Gamma(e^X).$$

The quantization theorem from Subsection 3.1 may be regarded as a generalized **Noether principle** that automatically promotes all symmetries of Z to dynamical symmetries of the corresponding quantum system.

Thus a coherent space contains intrinsically all information needed to interpret the quantum system, including that about which operators may be treated as q-observables.

The dynamics of a physical system is traditionally given by a **Hamiltonian**, a symmetric and Hermitian expression H in the q-observables. If the coherent space is in fact a **coherent manifold**, the **classical dynamics** determined by the Hamiltonian is given by a Poisson bracket canonically associated to the coherent space through variation of the so-called **Dirac–Frenkel action** discussed in Subsection 4.4 below. (Classical mechanics on Poisson

manifolds, the most general setting for the dynamics in closed classical systems, is discussed in detail in MARS DEN & RATIU [34]. Less general is classical mechanics on symplectic manifolds, and even more restricted is classical mechanics on cotangent bundles, which includes classical mechanics on phase space \mathbb{R}^{6N} for systems of N particles in Cartesian coordinates.)

In a classical Hamiltonian system, the dynamics of a phase space function f is given by $\dot{f} = H \angle f$ where $f \angle g = \{g, f\}$ in terms of the Poisson bracket. For an N -particle system with particle positions q_j and particle momenta p_j , specializing this to $f = q_j$ and $g = p_j$ gives the classical equations of motion. In a quantum system, one has the same in the Heisenberg picture, and according to every textbook, the resulting dynamics is equivalent to the Schrödinger equation in the Schrödinger picture.

Exactly solvable systems. In the special case where the classical Hamiltonian is an infinitesimal symmetry of Z , and hence the quantum Hamiltonian has the form $\Gamma(H)$, the quantization lifts the classical phase space trajectory to a quantum trajectory. Thus if the Lie algebra of q-observables contains the Hamiltonian (and in some slightly more general situations), the quantum dynamics has the special feature that coherence is dynamically preserved. In terms of the Hamiltonian, the dynamics for pure quantum states ψ is traditionally given by the **time-dependent Schrödinger equation**

$$i\hbar \frac{d\psi}{dt} = d\Gamma(H)\psi$$

for the corresponding **quantum Hamiltonian** $d\Gamma(H)$. A dynamical symmetry preserved by H (in the classical case) or $d\Gamma(H)$ (in the quantum case) is a true symmetry of the corresponding classical or quantum system. The Fourier transform $\widehat{\psi}(E)$ satisfies the **time-independent Schrödinger equation**

$$d\Gamma(H)\widehat{\psi} = E\widehat{\psi}. \tag{12}$$

The following result shows that the solution of Schrödinger equations with a sufficiently nice Hamiltonian can be reduced to solving differential equations on Z .

3.1 Theorem. *Let Z be a coherent space, and \mathbb{G} be a Lie group of coherent maps with associated Lie algebra \mathbb{L} . Let $H(t) \in \mathbb{L}$ be a Hamiltonian with possibly time-dependent coefficients. Then the solution of the initial value problem*

$$i\hbar \frac{\partial}{\partial t} \psi_t = d\Gamma(H(t))\psi_t, \quad \psi_0 = |z_0\rangle \tag{13}$$

with $z_0 \in Z$ has for all times $t \geq 0$ the form of a coherent state, $\psi_t = |z(t)\rangle$ with the trajectory $z(t) \in Z$ defined by the initial value problem

$$i\hbar \dot{z}(t) = H(t)z(t), \quad z(0) = z_0.$$

This means that if a system is at some time in a coherent state it will be at all times in a coherent state.

This conservation of coherence has the consequence that the quantum system is **exactly solvable**. This means that the complete solution of the dynamics of the quantum system can be reduced to the solution of the corresponding classical system. Effectively, the partial differential equations of quantum mechanics in the quantum space are solved in terms of ordinary differential equations on the underlying coherent space. In many cases, this implies that the spectrum can be determined explicitly in terms of the representation theory of the corresponding Lie algebras.

More generally (see, e.g., IACHELLO [24]), we have an exactly solvable system whenever the Hamiltonian $H(t)$ is a linear combination of infinitesimal symmetries with coefficients given by **Casimirs** of the Lie algebra \mathbb{L} of infinitesimal symmetries, i.e., in the classical case central elements of the Lie–Poisson algebra $C^\infty(\mathbb{L}^*)$, and in the quantum case of the universal enveloping algebras of \mathbb{L} . On any orbit of the symmetry group, these Casimirs are represented by multiplication with a constant. One can therefore extend the coherent space Z without changing the quantum space by treating the corresponding multiples of the coherent states as new coherent states of an extended coherent space whose elements are labelled by pairs of elements of Z and appropriate multipliers. This turns the algebra of Casimirs into an abelian group of symmetries of the extended coherent space, which, together with original symmetries provides an action of a central extension of the original symmetry group as a symmetry group of the extended coherent space.

The time-independent Schrödinger equation (12) generalizes easily to a more general **implicit Schrödinger equation** $I(E)\psi = 0$. This more general formulation fits naturally the coherent space setting, and everything said so far (corresponding to $I(E) = E - d\Gamma(H)$) generalizes to the general implicit formulation.

3.3 Dynamical Lie algebras

A quantum dynamical problem can often be reduced to finding the spectrum of a physical system defined by an implicit Schrödinger equation

$$I(E)\psi = 0 \tag{14}$$

with an energy-dependent **system operator** $I(E)$, and ψ in the antidual of some Euclidean space \mathbb{H} . A nonlinear $I(E)$ typically appears in reduced effective descriptions of systems derived from a more complicated Hamiltonian setting and in relativistic systems. (The antidual is needed to account for a possible continuous spectrum.)

This section discusses implicit Schrödinger equations for the exactly solvable case where the system operator $I(E)$ is contained in a Lie algebra \mathbb{L} with known representation theory.

This is the setting where a tractable dynamical symmetry group for the Hamiltonian is known and covers many interesting systems.

For example, the system operator $I(E) = p_0^2 - \mathbf{p}^2 - (mc)^2$ with $p_0 = E/c$ describes a free spin 0 particle. This generalizes to a quadratic implicit Schrödinger equation

$$\left(\pi^2 - \frac{ige\hbar}{c}\mathbf{S} \cdot \mathbf{F}(x) - (mc)^2\right)\psi = 0 \quad (15)$$

for a particle of charge e , mass m , and arbitrary spin in an electromagnetic field. Here

$$\pi = \begin{pmatrix} \pi_0 \\ \boldsymbol{\pi} \end{pmatrix} := p + e\hbar A(x) \quad (16)$$

is a gauge invariant 4-vector, \mathbf{S} is the 3-dimensional spin vector representing the intrinsic angular momentum of a particle of spin $j = 0, \frac{1}{2}, 1, \dots$, the 3-vector $\mathbf{F}(x) = \mathbf{E}(x) + ic\mathbf{B}(x)$ is the Riemann–Silberstein vector encoding the electric field $\mathbf{E}(x)$ and the magnetic field $\mathbf{B}(x)$, and g is the dimensionless **g-factor** of the **magnetic moment**

$$\boldsymbol{\mu}_s := -\frac{g\mu_B}{\hbar}\mathbf{S},$$

where μ_B is a constant called the **Bohr magneton**, and ψ is a wave function with $s = 2j + 1$ components. For spin $j = 1/2$, we have $s = 2$ components, hence, being second order, 4 local degrees of freedom, corresponding to the 4 components of the (first order) Dirac equation, which is equivalent to the special case $g = 2$.

In the special case where the dependence on E is linear, we have

$$I(E) = EM - N \quad (17)$$

with fixed $M, N \in \mathbb{L}$. This covers the simple case of a harmonic oscillator, where $M = 1$, $N = \frac{1}{2}(p^2/m + Kq^2)$ is the Hamiltonian, and the Lie algebra is the **oscillator algebra**, with generators $1, p, q, H$ (or, in complex form, $1, a, a^*, a^*a$). It also covers a family of practically relevant exactly solvable systems with Lie algebra $\mathbb{L} = so(2, 1) \oplus \mathbb{C} = su(1, 1) \oplus \mathbb{C}$ discussed in detail in the book by WYBOURNE [57], containing among others the case of a particle of mass m in a Coulomb field, with $M = r = |\mathbf{q}|$ and $N = MH$, where

$$H = \frac{1}{2}m\mathbf{v}^2 - \frac{\alpha}{|\mathbf{q}|}$$

is the Coulomb Hamiltonian.

If $I(E)$ belongs for all E to some Lie algebra \mathbb{L} acting on \mathbb{H} in a (reducible or irreducible) representation then \mathbb{L} is called a **dynamical Lie algebra**² of the problem.

²One can always take the dynamical Lie algebra to be the Lie algebra $\text{Lin } \mathbb{H}$ of all linear operators on the nuclear space \mathbb{H} . For this choice, the dynamical Lie algebra offers no advantage over the standard treatment. Therefore it is usually understood that the dynamical Lie algebra is much smaller than $\text{Lin } \mathbb{H}$, although mathematically there is no such restriction.

In general, the requirement for a dynamical symmetry group is just that all quantities of physical interest in the system can be expressed in the Lie–Poisson algebra (in the classical case) or the universal enveloping algebra (in the quantum case) of the corresponding Lie algebra. In this case, the label “dynamical” is a misnomer, and **kinematic symmetry group** would be more appropriate. The kinematic symmetry group is an integral part of the Hamiltonian or Lagrangian setting; so one usually gets it directly from the formulation and a look at the obvious symmetries. For any anharmonic oscillator it is $Sp(2)$; for any system of N particles in \mathbb{R}^3 it is the symplectic group $Sp(6N)$, generated by the inhomogeneous quadratics in p and q .

If a problem has a dynamical symmetry group such that the (discrete or continuous) spectrum of all elements of its Lie algebra \mathbb{L} is exactly computable then the spectrum of the system can be found exactly. In the best understood cases, \mathbb{L} is a finite-dimensional semisimple Lie algebra. Here everything is tractable more or less explicitly since the representation theory of these Lie algebras and their corresponding groups is fully understood. A problem solvable in this way is called **integrable**.

The **spectrum** of the nonlinear eigenvalue problem (14) is the set $\text{Spec } I$ of all $E \in \mathbb{C}$ such that $I(E)$ is not invertible. In terms of (generalized) eigenvalues and eigenvectors of $I(E)$,

$$I(E)|\xi, E\rangle = \lambda(\xi, E)|\xi, E\rangle,$$

where ξ is a label distinguishing different eigenvectors $|\xi, E\rangle$ in a (generalized) orthonormal basis of the eigenspace corresponding to the eigenvalue E . To cover the continuous spectrum (where eigenvectors are unnormalized, hence do not belong to the Hilbert space), we work in a Euclidean space \mathbb{H} on which the Hamiltonian acts as a linear operator. The Hilbert space of the problem is then the completion $\overline{\mathbb{H}}$ of this space, and $\mathbb{H} \subseteq \overline{\mathbb{H}} \subseteq \mathbb{H}^\times$ is a Gelfand triple. Therefore

$$I(E)|\xi, E\rangle = 0 \quad \text{whenever } \lambda(\xi, E) = 0.$$

Thus

$$\text{Spec } I = \{E \in \mathbb{R} \mid \lambda(\xi, E) = 0 \text{ for some } \xi \in \text{Spec } I(E)\}$$

Moreover, it is easy to see that all eigenvectors of the nonlinear eigenvalue problem have the form $|\xi, E\rangle$. Thus the spectrum is given by the set of solutions of the nonlinear equation $\lambda(\xi, E) = 0$.

In many cases of interest (e.g., cf. Subsection 4.1, when \mathbb{L} is a Lie $*$ -algebra), $\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{C}$; then we may write

$$I(E) := m(E)X(E) - k(E), \tag{18}$$

where $m(E)$ and $k(E)$ are scalars not vanishing simultaneously, and $X(E) \in \mathbb{L}_0$. If

$$X(E)|\xi, E\rangle = \xi|\xi, E\rangle$$

is a complete system of (generalized) eigenvalues and eigenvectors of $X(E)$ then

$$I(E)|\xi, E\rangle = \lambda(\xi, E)|\xi, E\rangle, \quad \lambda(\xi, E) = m(E)\xi - k(E). \quad (19)$$

Therefore

$$I(E)|\xi, E\rangle = 0 \quad \text{whenever } \lambda(\xi, E) = 0.$$

Again, all eigenvectors of the nonlinear eigenvalue problem have the form $|\xi, E\rangle$, and the spectrum is given by the set of solutions of $\lambda(\xi, E) = 0$.

If a problem has a dynamical symmetry group such that the (discrete or continuous) spectrum of all elements of its Lie algebra \mathbb{L} is exactly computable then the spectrum of the system can be found exactly. In the best understood cases, \mathbb{L} is a finite-dimensional semisimple Lie algebra. Here everything is tractable more or less explicitly since the representation theory of these Lie algebras and their corresponding groups is fully understood. In this case, one may find the $|s, E\rangle$ by transforming $I(E)$ to elements from a standard set of representatives of the conjugacy classes, and has to work out explicit spectral factorizations for these. For semisimple Lie algebras \mathbb{L} in finite dimensions, each Lie algebra element is in a Cartan subalgebra, and the latter are all unitarily conjugate, i.e., if V and V' are Cartan subalgebras, there is a group element U such that $V' = \{\text{ad}_U X \mid X \in V\}$. So one only has to consider conjugacy inside the standard Cartan subalgebra. (In the noncompact case, the eigenvectors correspond to representatives from any conjugacy class, which may be several in the same irreducible representation) This is enough to give the spectrum, and in the discrete case the full spectral resolution. In the continuous case, one still needs to find the spectral density and from it the S-matrix; cf. KERIMOV [27, 28].

3.4 Relations to geometric quantization

Often, classical symmetries are promoted to quantum symmetries in a projective representation. Then the symmetry group of the extended phase space is a proper central extension of the symmetry group of the original space. It acts on an extended phase space whose dimension is 1 larger. For example, the classical phase space with n spatial degrees of freedom has dimension $2n$, but the associated Heisenberg algebra, the central extension of an abelian group with $2n$ generators, has dimension $2n + 1$, as the canonical commutation relations for the extended Poisson bracket (or in the quantum case for the commutator) require an additional central generator.

Such a central extension is the rule rather than the exception. The extra dimension, often called **Berry phase** or **geometric phase**, accounts for topological features such as the Aharonov–Bohm effect. But it also occurs in classical physics; e.g., a classical electromagnetic field exhibits topological effects when the field strength is not globally integrable to a vector potential.

The explicit description of a central extension in terms of the original symmetry group involves so-called **cocycles**. Rather than with the original symmetry group, one can indeed work directly with a central extension of the group, acting on the extended phase space. (Examples where this works are the Möbius space and the Klauder spaces discussed before.) In this way, one can avoid the use of cocycles, as the relevant projective representations become ordinary representations of the central extension. Thus the extended description generally reflects the quantum properties in a more symmetric way than the original coherent space.

In our present setting, the extended phase space is modeled by a projective coherent space. A **projective coherent space** is a coherent space with a **scalar multiplication** that assigns to each nonzero complex number λ and each $z \in Z$ a point $\lambda z \in Z$ such that

$$(C4) \quad \overline{\lambda z} = \overline{\lambda} \overline{z}, \quad \lambda(\mu z) = (\lambda\mu)z ;$$

$$(C5) \quad K(\lambda z, z') = \lambda^e K(z, z')$$

for some nonzero integer e . Projectivity is typically needed when one wants to have all symmetries of interest represented coherently. Projective coherent spaces coherently represent central extensions of groups in cases where the original group is represented by a projective representation that would lead to coherent maps only up to additional scalar factors called cocycles.

Geometrically, the extended phase space takes the form of a line bundle. In case of the Heisenberg algebra, the line bundle is trivial, formed by $Z = \mathbb{C} \times \mathbb{C}^n$ with componentwise conjugation, scalar multiplication defined by $\alpha(\lambda, s) := (\alpha\lambda, s)$, and coherent product

$$K(z, z') := \lambda\lambda' e^{s^T s' / \hbar} \quad \text{for } z = (\lambda, s), \quad z' = (\lambda', s')$$

we get a projective coherent space Z whose quantum space $\mathbb{Q}(Z)$ is the bosonic Fock space with n independent oscillators, and the coherent states are the multiples of the Glauber coherent states. Indeed, the coherent states

$$|\lambda, s\rangle, \quad \lambda, s \in \mathbb{C}$$

in a single-mode Fock space have the Hermitian inner product

$$\langle \lambda, s | \lambda', s' \rangle = \overline{\lambda}\lambda' e^{\overline{s}s' / \hbar}.$$

By means of Klauder spaces (defined above), the construction easily extends to an arbitrary finite or infinite number of modes. In terms of the traditional Fock space description, the coherent states are the simultaneous eigenstates of the annihilator operators,

$$a|z\rangle = \mathbf{z}|z\rangle \quad \text{for } z \in Z.$$

More generally (see NEUMAIER & GHAANI FARASHAHI [41]), Klauder spaces provide an elegant and efficient approach to the properties of creation and annihilation operators.

A coherent space generalizes finite-dimensional symplectic manifolds with a polarization that induces a complex structure on the manifold. A projective coherent space generalizes a corresponding **Hermitian line bundle** Z , i.e., a line bundle with a Hermitian connection. Such line bundles are usually discussed in the context of geometric quantization.

Geometric quantization (see, e.g., BAR-MOSHE & MARINOV [5], ENGLIŠ [13, 14], SCHLICHENMAIER [51]) proceeds from a symplectic manifold \mathbb{K} . It constructs (in the group case in terms of integral³ cohomology) a polarization that defines a Hermitian line bundle $Z = \mathbb{C}\mathbb{K}$ and an associated **Kähler potential** (which is essentially the logarithm of the coherent product). This potential turns \mathbb{K} into a Kähler manifold with a natural Kähler metric, Kähler measure, and symplectic Kähler bracket. If the Kähler metric is definite (which is always the case if Z is a compact symmetric space), there is an associated Hilbert space of square integrable functions on which quantized operators can be defined by a recipe of VAN HOVE [23].

An **involutive coherent manifold** is a coherent manifold Z equipped with a smooth mapping that assigns to every $z \in Z$ a **conjugate** $\bar{z} \in Z$ such that $\overline{\bar{z}} = z$ and $\overline{K(z, z')} = K(\bar{z}, \bar{z}')$ for $z, z' \in Z$. Under additional conditions, an involutive coherent manifold carries a canonical Kähler structure turning it into a **Kähler manifold**. For semisimple finite-dimensional Lie algebras, the irreducible highest weight representations have nice coherent space formulations. In the literature, the logarithm of the coherent product figures under the name of Kähler potential. ZHANG et al. [59] relate the latter to coherent states. The coherent quantization of Kähler manifolds is equivalent to traditional **geometric quantization of Kähler manifolds**. But in the coherent setting, quantization is not restricted to finite-dimensional manifolds, which is important for quantum field theory.

The coherent product and the conjugation are C^∞ -maps on the line bundle. In order that this line bundle exists, the symplectic manifold must also carry a positive definite Kähler potential $F : Z \times Z \rightarrow \mathbb{C}$ satisfying a generalized **Bohr–Sommerfeld quantization condition** defined by the integrality of some cohomological expression. In this case, Z is a projective coherent space with coherent product

$$K(z, z') := e^{-F(z, z')}.$$

The quantum space of the projective coherent space carries the representation satisfying the conditions of a successful geometric quantization. This procedure, called **Berezin quantization**, is the most useful way of performing geometric quantization; see, e.g., SCHLICHENMAIER [51].

³Integral cohomology apparently corresponds to the fact that the line bundle can in fact be viewed as a $U(1)$ -bundle so that phases are well-defined.

4 The thermal interpretation in terms of Lie algebras

Coherent quantum physics on a coherent space Z is related to physical reality by means of the thermal interpretation, discussed in detail in Part II [43] and applied to measurement in Part III [44] of this series of papers. We rephrase the formal essentials of the thermal interpretation in a slightly generalized more abstract setting, to emphasize the essential mathematical features and the close analogy between classical and quantum physics. We show how the coherent variational principle (the Dirac–Frenkel procedure applied to coherent states) can be used to show that in coarse-grained approximations that only track a number of relevant variables, quantum mechanics exhibits chaotic behavior that, according to the thermal interpretation, is responsible for the probabilistic features of quantum mechanics.

4.1 Lie *-algebras

A (complex) **Lie algebra** is a complex vector space \mathbb{L} with a distinguished **Lie product**, a bilinear operation on \mathbb{L} satisfying $X \angle X = 0$ for $X \in \mathbb{L}$ and the **Jacobi identity**

$$X \angle (Y \angle Z) + Y \angle (Z \angle X) + Z \angle (X \angle Y) = 0 \quad \text{for } X, Y, Z \in \mathbb{L}.$$

A **Lie *-algebra** is a complex Lie algebra \mathbb{L} with a distinguished element $1 \neq 0$ called **one** and a mapping $*$ that assigns to every $X \in \mathbb{L}$ an **adjoint** $X^* \in \mathbb{L}$ such that

$$(X + Y)^* = X^* + Y^*, \quad (X \angle Y)^* = X^* \angle Y^*,$$

$$X^{**} = X, \quad (\lambda X)^* = \lambda^* X^*,$$

$$1^* = 1, \quad X \angle 1 = 0$$

for all $X, Y \in \mathbb{L}$ and $\lambda \in \mathbb{C}$ with complex conjugate λ^* . We identify the multiples of 1 with the corresponding complex numbers.

A **state** on a Lie *-algebra \mathbb{L} is a positive semidefinite Hermitian form $\langle \cdot, \cdot \rangle$, antilinear in the first argument and normalized such that $\langle 1, 1 \rangle = 1$.

A group G **acts** on a Lie *-algebra \mathbb{L} if for every $A \in G$, there is a linear mapping that maps $X \in \mathbb{L}$ to $X^A \in \mathbb{L}$ such that

$$(X \angle Y)^A = X^A \angle Y^A,$$

$$(X^A)^B = X^{AB}, \quad (X^A)^* = (X^*)^A, \quad X^1 = X, \quad 1^A = 1$$

for all $X, Y \in \mathbb{L}$ and all $A, B \in G$. Thus the mappings $X \rightarrow X^A$ are *-automorphisms of the Lie *-algebra. Such a family of mappings is called a **unitary representation** of G on \mathbb{L} .

Often, unitary representations arise by writing the Lie $*$ -algebra \mathbb{L} as a vector space of complex $n \times n$ matrices closed under conjugate transposition $*$ and commutation, with $X \angle Y := \frac{i}{\hbar}[X, Y]$, and G as a group of unitary $n \times n$ matrices such that $X^A := A^{-1}XA \in \mathbb{L}$ for all $X \in \mathbb{L}$.

4.2 Quantities, states, uncertainty

In classical and quantum physics, **physical systems** are modeled by appropriate Lie $*$ -algebras \mathbb{L} , whose elements are interpreted as the **quantities** of the system modeled. Each physical system may exist in different **instances**; each instance specifies a particular system under particular conditions. A state defines the **properties** of an instance of a physical system described by a model, and hence what **exists** in the system. Properties depend on the state and are expressed in terms of definite but uncertain values of the quantities:

(GUP) General uncertainty principle: *In a given state, any quantity $X \in \mathbb{L}$ has the uncertain value*

$$\bar{X} := \langle X \rangle := \langle 1, X \rangle \quad (20)$$

with an **uncertainty** of⁴

$$\sigma_X := \sqrt{\langle X - \bar{X}, X - \bar{X} \rangle} = \sqrt{\langle X, X \rangle - |\bar{X}|^2}. \quad (21)$$

Through (20), each state induces an element $\langle \cdot \rangle$ of the **dual** of \mathbb{L} , the space \mathbb{L}^* of linear functionals on \mathbb{L} .

As discussed in Part I [42] (and exemplified in more detail in Part III [44] and Part IV [45]), the **interpretation**, i.e., the identification of formal properties given by uncertain values with real life properties of a physical system, is done by means of

(CC) Callen's criterion (CALLEN [11, p.15]): *Operationally, a system is in a given state if its properties are consistently described by the theory for this state.*

This is enough to find out in each single case how to approximately measure the uncertain value of a quantity of interest, though it may require considerable experimental ingenuity to do so with low uncertainty. The uncertain value \bar{X} is considered informative only when its uncertainty σ_X is much less than $|\bar{X}|$.

As position coordinates are dependent on a convention about the coordinate system used, so all system properties are dependent on the **conventions** under which they are viewed.

⁴Since the state is positive semidefinite, the first expression shows that σ_X is a nonnegative real number. The equivalence of both expressions defining σ_X follows from $\langle X \rangle = \bar{X}$ and

$$\langle X - \bar{X}, X - \bar{X} \rangle = \langle X, X \rangle - \langle X, \bar{X} \rangle - \langle \bar{X}, X \rangle + \langle \bar{X}, \bar{X} \rangle = \langle X, X \rangle - \langle X \rangle^* \bar{X} - \bar{X}^* \langle X \rangle + |\bar{X}|^2 = \langle X, X \rangle - |\bar{X}|^2.$$

To be objective, these conventions must be interconvertible. This is modeled by a group G of **symmetries** acting transitively both on the spacetime manifold M considered and on the set W of conventions. We write these actions on the left, so that $A \in G$ maps $x \in M$ to Ax and $w \in W$ to Aw .

To be applicable to a physical system, a representation of G on the Lie $*$ -algebra \mathbb{L} of quantities must be specified. Depending on the model, this representation accounts for conservative dynamics and the principle of relativity in its nonrelativistic, special relativistic, or general relativistic situation. It also caters for the presence of internal symmetries of a physical system. Correspondingly, G may be a group of matrices, a Heisenberg group, the Galilei group, the Poincaré group, or a group of volume-preserving diffeomorphisms of a spacetime manifold M .

A **particular physical system** in all its views is described by a family of states $\langle \cdot, \cdot \rangle_w$ indexed by a convention $w \in W$ satisfying the covariance condition

$$\langle X, Y \rangle_{Aw} = \langle X^A, Y^A \rangle_w \quad (22)$$

for $X, Y \in \mathbb{L}$, $w \in W$, $A \in G$. In particular, uncertain values transform as

$$\langle X \rangle_{Aw} = \langle X^A \rangle_w. \quad (23)$$

A **subsystem** of a particular physical system is defined by specifying a Lie $*$ -subalgebra and restricting the family of states to this subalgebra.

If (as is commonly done) we work within a fixed affine coordinate system in a spacetime (homeomorphic to some) \mathbb{R}^d , the only conditions relevant are when and where a system is described; all other conditions are handled implicitly by covariance considerations. In this case, W is simply the spacetime M , and G is the group of affine translations $T_z : x \rightarrow x + z$ of M by z . In this case,

$$\bar{X}(x) = \langle X \rangle_x, \quad \sigma_X(x) = \sqrt{\langle X, X \rangle_x - |\bar{X}(x)|^2}$$

define the **value** $\bar{X}(x)$ of X at x and its **uncertainty** $\sigma_X(x)$ at x , and (22) and (23) become

$$\langle X, Y \rangle_{x+z} = \langle X^{Tz}, Y^{Tz} \rangle_x, \quad \langle X \rangle_{x+z} = \langle X^{Tz} \rangle_x.$$

The value $\bar{X}(x)$ is (in principle) **observable** with **resolution** $\delta > 0$ if it varies slowly with x and has a sufficiently small uncertainty. More precisely, if Δ denotes the set of spacetime shifts that are imperceptible in the measurement context of interest, observability with resolution δ requires that

$$|\bar{A}(x+h) - \bar{A}(x)| \leq \delta \quad \text{for } h \in \Delta,$$

$$\sigma_X(x) \ll |\bar{A}(x)| + \delta.$$

We require that the translation group is generated by a covariant **momentum vector** $p \in \mathbb{L}^d$ with Hermitian components, in the sense that

$$\frac{\partial}{\partial x_\nu} X^{T_x} = p_\nu \angle X \quad (24)$$

for $X \in \mathbb{L}$, $x \in M$ and all indices ν . From the covariance condition (22), we conclude that

$$\frac{\partial}{\partial x_\nu} \langle X, Y \rangle_x = \langle p_\nu \angle X, Y \rangle_x + \langle X, p_\nu \angle Y \rangle_x. \quad (25)$$

In particular, the uncertain values satisfy the **covariant Ehrenfest equation**

$$\frac{\partial}{\partial x_\nu} \langle X \rangle_x = \langle p_\nu \angle X \rangle_x \quad (26)$$

discussed in a special case in Part II [43].

In classical or quantum multiparticle mechanics (as opposed to field theory), space and time are treated quite differently, and we are essentially in the case $d = 1$ of the above, where the convention about views of system properties is completely specified by the time $t \in \mathbb{R}$. In this case, the above specializes to

$$\bar{X}(t) = \langle X \rangle_t, \quad \sigma_X(t) = \sqrt{\langle X, X \rangle_t - |\bar{X}(t)|^2}$$

The time translation group is generated by a Hermitian **Hamiltonian** $H \in \mathbb{L}$, and

$$\frac{d}{dt} X^{T_t} = H \angle X. \quad (27)$$

$$\frac{\partial}{\partial t_\nu} \langle X, Y \rangle_t = \langle H \angle X, Y \rangle_t + \langle X, H \angle Y \rangle_t. \quad (28)$$

In particular, the uncertain values satisfy the **Ehrenfest equation**

$$\frac{d}{dt} \langle X \rangle_t = \langle H \angle X \rangle_t, \quad (29)$$

providing a deterministic dynamics for the q-expectations.

4.3 Examples

1. A simple classical example is $\mathbb{L} = \mathbb{C}^3$ with the cross product as Lie product. It is isomorphic to the Lie algebra $so(3, \mathbb{C})$ and describes in this representation a rigid rotator. The dual space \mathbb{L}^* is spanned by the three components of J , and the functions of J^2 are the

Casimir operators. Assigning to J a particular 3-dimensional vector with real components (since J has Hermitian components) gives the classical angular momentum in a particular state.

2. The same Lie algebra is also isomorphic to $su(2)$, the Lie algebra of traceless Hermitian 2×2 matrices, and then describes the thermal setting of a single qubit. In this case, we think of \mathbb{L}^* as mapping the three Hermitian Pauli matrices σ_j to three real numbers S_j , and extending the map linearly to the whole Lie algebra. Augmented by $S_0 = 1$ to account for the identity matrix, which extends the Lie algebra to that of all Hermitian matrices, this leads to the classical description of the qubit discussed in Subsection 3.5 of Part III [44].

3. Consider the Lie $*$ -algebra \mathbb{L} of smooth functions $f(p, q)$ on classical phase space with the negative Poisson bracket as Lie product and $*$ as complex conjugation. Given a $*$ -homomorphism ω with respect to the associative pointwise multiplication, determined by the classical values $\bar{p}_k := \omega(p_k)$ and $\bar{q}_k := \omega(q_k)$, the states defined by

$$\langle X, Y \rangle := \omega(X^*Y)$$

reproduce classical deterministic dynamics. More generally, \mathbb{L} can be partially ordered by defining $f \geq 0$ iff f takes values in the nonnegative reals. Given a monotone $*$ -linear functional ω on \mathbb{L} satisfying $\omega(X^*) = \omega(X)^*$ for $X \in \mathbb{L}$ and $\omega(1) = 1$, the states defined by

$$\langle X, Y \rangle := \omega(X^*Y)$$

reproduce classical stochastic dynamics in the Koopman picture discussed in Subsection 4.1 of Part III [44]. In both cases, $\langle X \rangle = \omega(X)$.

4. The basic example of interest for isolated quantum physics is the Lie $*$ -algebra $\mathbb{L}(Z)$ of linear operators acting on the quantum space $\mathbb{H} = \mathbb{Q}(Z)$ of the coherent space Z , with Lie product

$$X \angle B := \frac{i}{\hbar}[X, B] = \frac{i}{\hbar}(XB - BX). \quad (30)$$

The action of the translation group on $X \in \mathbb{L}$ is given by

$$X^{T_x} := U(x)^* X U(x)$$

with unitary operators $U(x)$ satisfying $U(0) = 1$ and $U(x)U(y) = U(x + y)$. The states of interest are the **regular states**, defined by

$$\langle X, Y \rangle_x = \text{Tr}(Y \rho(x) X^*)$$

for some positive semidefinite Hermitian **density operator** $\rho(x) \in \overline{\mathbb{Q}}(Z)$ with $\text{Tr} \rho(x) = 1$. In this case, the uncertain values

$$\langle X \rangle_x = \text{Tr}(X \rho(x))$$

viewed from $x \in \mathbb{R}^d$ are the **q-expectations**⁵ of X , and the uncertainty can be expressed of q-expectations, too. For Hermitian X , it is given by

$$\sigma_X(x) = \sqrt{\langle X^2 \rangle_x - \bar{X}(x)^2}.$$

4.4 The coherent variational principle

A basic principle is the Dirac–Frenkel approach for reducing a nonrelativistic quantum problem to an associated classical approximation problem. In this approach, the variational principle for classical Lagrangian systems is rewritten for the present situation and then called the **Dirac–Frenkel variational principle**. It was first used by DIRAC [12] and FRENKEL [17], and found numerous applications; a geometric treatment is given in KRAMER & SARACENO [33]. The action takes the form

$$I(\psi) = \int dt \psi^* (i\hbar\partial_t - H)\psi = \int dt (i\hbar\psi^*\dot{\psi} - \psi^*H\psi) \quad (31)$$

where the **quantum Hamiltonian** $H \in \text{Lin}^\times \mathbb{H}$ is a self-adjoint operator. The coherent 1-form θ may be interpreted as the Lagrangian 1-form corresponding to the Dirac–Frenkel action. The Legendre transform of the **Lagrangian**

$$L(\psi) := i\hbar\psi^*\dot{\psi} - \psi^*H\psi$$

is the corresponding classical Hamiltonian

$$\langle H \rangle = \psi^*H\psi.$$

The Dirac–Frenkel action is stationary iff ψ satisfies the **Schrödinger equation**

$$i\hbar\dot{\psi} = H\psi,$$

If one has a coherent space Z and $\mathbb{H} = \mathbb{Q}(Z)$ a quantum space of Z , one can restrict ψ to coherent states, and we get an action

$$I(z) = \int dt \langle z | (i\hbar\partial_t - H) | z \rangle$$

for the path $z(t)$. This coherent variational principle has first been proposed by KLAUDER [29]. The variational principle for the action $I(z)$ defines an approximate classical Lagrangian (and hence conservative) dynamics for the parameter vector $z(t)$. This **coherent dynamics** on Z is regarded as a semiclassical (or **semiquantal**) approximation of the

⁵Traditionally, $\langle X \rangle$ is called the expectation value of X , but such a statistical interpretation is not needed, and is not even possible when X has no spectral resolution.

quantum dynamics. In two important cases, the norm of the state is preserved by the coherent dynamics – Z must either be **normalized**, i.e., $K(z, z) = 1$ for all $z \in Z$, or projective (as defined in Subsection 3.4). The approximation turns out to be exact when the Hamiltonian belongs to the infinitesimal Lie algebra of the symmetry group of the coherent state. It is inexact but good if it is not too far from such an element.

The classical problem created by the Dirac–Frenkel approach is again conservative, based on a classical action, which may or may not be transformable into an equivalent Hamiltonian problem. The latter depends on whether the Dirac–Frenkel Lagrangian is regular or singular. Thus it is important that one understands the structure of classical singular Lagrangian problems.

4.5 Coherent numerical quantum physics

The Dirac–Frenkel variational principle is the basis of much of traditional numerical quantum mechanics, which heavily relies on variational methods. It plays an important role in approximation schemes for the dynamics of quantum systems. In many cases, a viable approximation is obtained by restricting the state vectors $\psi(t)$ to a linear or nonlinear manifold of easily manageable states $|z\rangle$ parameterized by classical parameters z which can often be given a physical meaning.

What is commonly called a mean field theory is just the simplest coherent state approximation. This is already much better than a classical limit view, and in particular corrects for the missing zero point energy terms in the latter.

An important application of this situation are the **time-dependent Hartree–Fock equations** (see, e.g., MCLACHLAN & BALL [35]), obtained by choosing Z to be a Grassmann space.⁶ This gives the Hartree–Fock approximation, which is at the heart of dynamical simulations in quantum chemistry. It can usually predict energy levels of molecules to within 5% accuracy. Choosing Z to be a larger space (obtained by the methods of Subsection 2.3) enables one to achieve accuracies approaching 0.001%.

Apart from Hartree–Fock calculations (symmetry group $U(N)$ on coadjoint orbits of Slater states), this covers Hartree–Fock–Bogoliubov methods (see, e.g., GOODMAN [18]), which include Bogoliubov transformations to get a quasiparticle picture (symmetry group $SO(2N)$), and Gaussian methods (see, e.g., PATTANAYAK & SCHIEVE [49], ONO & ANDO [47]) used in quantum chemistry (symmetry group $ISp(2N)$). There are time-dependent versions of these, and extensions that go beyond the mean field picture, using either Hill–Wheeler equations in the generator coordinate method (see, e.g., GRIFFIN & WHEELER [20]) or coupled cluster expansions (see, e.g., BARTLETT & MUSIAL [6]) around the mean field.

⁶A Grassmann space is a manifold of all k -dimensional subspaces of a vector space. It is one of the symmetric spaces.

4.6 Coherent chaos

Since the Dirac–Frenkel variational principle gives a reduced deterministic dynamics for $z(t)$, it fits in naturally with the thermal interpretation. As discussed in Subsection 4.2 of Part III [44], it is one of the ways to obtain a coarse-grained approximate dynamics for a set of relevant beable, in this case the $z \in Z$ labeling coherent states. Here we show that it can be used to study how – in spite of the linearity of the Schrödinger equation – chaos emerges through coarse-graining from the exact quantum dynamics.

ZHANG & FENG [58] used the Dirac–Frenkel variational principle restricted to general coherent states to get a semiclassical system of ordinary differential equations approximating the dynamics of the q-expectations of macroscopic operators of certain multiparticle systems. At high resolution, this deterministic dynamics is highly chaotic. This chaoticity is a general feature of approximation schemes for the dynamics of q-expectations or the associated reduced density functions. In particular, as discussed in detail in Part III [44], this seems enough to enforce the probabilistic nature of microscopic measurements using macroscopic devices.

Zhang and Feng derive in a purely mathematical way – without referring to probability or statistics – the equations that they show to be chaotic. Thus what they do is completely independent of any particular interpretation of quantum physics. They construct a semiclassical dynamics (where the relevant operators are replaced by their q-expectations) and then discuss the resulting system of ordinary differential equations. It turns out to be chaotic. The exact quantum dynamics would be given instead by partial differential equations!

In the overview of their paper, ZHANG & FENG [58, pp.4–9] state that they focus attention on understanding the question of quantum-classical correspondence (QCC), the search for an unambiguous classical limit, starting purely from quantum theory. They explore how, under suitable conditions, classical chaos can emerge naturally from quantum theory. They use the semiclassical method discussed above for the exploration of the correspondence between quantum and classical dynamics as well as quantum nonintegrability. They mention the relations to geometric quantization and coherent states, and work in a group theoretic setting corresponding to coherent states defined by coadjoint orbits of semisimple Lie groups. Their coset space G/H (or rather a complex line bundle over it arising in geometric quantization and carrying some of the phase information) discussed in [58, pp.39] is a coherent space with coherent product given by their (3.1.8). The variation of the effective quantum action in their (3.2.11) is the Dirac–Frenkel variational principle. The result of the variation is a symplectic system of differential equations that has a semiclassical (or as they say, semiclassical) interpretation. This system gives an approximate dynamics for the q-expectations of the generators of the dynamical group. This dynamics is chaotic when the classical limit of the quantum system is not integrable.

5 Field theory

This section defines the meaning of the notion of a field in the abstract setting of Section 4 and shows how coherent spaces may be used to define relativistic quantum field theories. Nothing more than basic definitions and properties are given; details will be given elsewhere.

5.1 Fields

In the general framework of Section 4, a **field** is an element ϕ of the space of \mathbb{L} -valued distributions $\mathbb{L} \otimes S(M, V)^*$ satisfying

$$\frac{\partial}{\partial x_\nu} \phi(x) = p_\nu \lrcorner \phi(x). \quad (32)$$

Here $S(M, V)^*$ is the dual of the Schwartz space $S(M, V)$ of rapidly decaying smooth functions on M with values in V (or the space of \mathbb{L} -valued sections of a corresponding fiber bundle with generic fiber V). Thus the **smearred fields**

$$\phi(f) := \int_M dx f(x) \phi(x),$$

defined for arbitrary test functions $f \in S(M, V)$, provide quantities in \mathbb{L} . The primary beables are the distribution-valued q-expectations

$$\phi_{\text{cl}}(x) = \langle \phi(x) \rangle_0$$

of fields and the distribution-valued **Greens functions**

$$W(x, y) = \langle \phi(x), \phi(y) \rangle_0$$

of field products at some fixed spacetime origin 0. After smearing with test functions, these distributions produce proper beables. A comparison of (32) with (24) shows that

$$\phi(x + z) = \phi(x)^{T_z}.$$

As a consequence, field expectations from different spacetime views satisfy

$$\langle \phi(x) \rangle_z = \langle \phi(x + h) \rangle_{z-h},$$

showing that the choice of a fixed origin is inessential; a change of origin only amounts to a spacetime translation.

We also see that for any quantity $A \in \mathbb{L}$, the definition

$$A(x) := A^{T_x} \quad \text{for } x \in M$$

defines a field. These fields are more regular than the fields occurring in relativistic quantum field theory, which are proper distributions.

5.2 Coherent spaces for quantum field theory

The techniques of geometric quantization do not easily extend (except on a case by case basis) to the quantization of infinite-dimensional manifolds, which would be necessary for modeling quantum field theories. However, the coherent space approach extends to quantum field theory. The coherent manifolds are now infinite-dimensional, and their topology is more technical to cope with than in the finite-dimensional case. The process of second quantization is such an example of quantization of infinitely many degrees of freedom. Thus second quantized calculations become tractable via infinite-dimensional coherent spaces. For example the calculus of creation and annihilation operators was developed in NEUMAIER & GHAANI FARASHAHI [41] in terms of Klauder spaces, giving simple proofs of many standard results on calculations in Fock spaces.

The groups that can be most easily quantized are infinite-dimensional analogues of the symplectic, orthogonal and (for fixed particle number) unitary groups, Kac–Moody groups, some related groups, and their abelian extensions. For example, the homogeneous quadratic expressions in finitely many creation and annihilation operators form a symplectic Lie algebra in the Boson case (CCR) and an orthogonal Lie algebra in the Fermion case (CAR); see ZHANG et al. [59].

This explains why knowing the representation theory of these groups (in the form of implications for their coherent spaces) is important.

Free quantum field theories are essentially the large N limit of the finite case. Large N amounts to discretizing configuration space or momentum space, keeping only N degrees of freedom. This is the basis of lattice methods. The thermodynamic limit $N \rightarrow \infty$ creates convergence problems – one has to struggle to avoid undefined expressions producing the infamous "infinities". The correct way to do this requires some functional analysis and introduces cocycles (that, for finite N , are trivial and hence can be avoided). For actual calculations (by computer), one needs everything as explicitly as possible, and coherent spaces yield explicit formulas for the things of interest.

In quantum field theory one needs to take the limit analytically rather than numerically, and a key problem is to decide when these limits exists and whether one can find them explicitly enough to get useful conclusions. Using these formulas allows one to replace the usual long-winded calculations with operators in the second-quantized formalism by fairly short arguments.

Some work on infinite-dimensional versions is available; in particular, for Fermions one needs the spin representation of infinite-dimensional orthogonal groups, constructed in terms of Pfaffians. The paper GRACIA-BONDÍA & VÁRILLY [19], though not very readable, contains lots of details (but not in terms of coherent spaces), and shows that the representation theory is enough to settle the case of QED in an external field. This is easier than full QED since

the field equations are linear. The mathematical challenge is the extension to nonlinear fields. (In [19], applications to the nonlinear case are promised for a follow-up paper, but I could not find any such paper.) In QED proper, asymptotic electrons are infraparticles rather than standard massive particles. Though we do not have a conventional Fock space, the asymptotic structure of QED is reasonably well understood. See, e.g., the work by HERDEGEN [21, 22] and KAPEC et al. [26].

Measures in infinite dimensions. For the quantization of infinite-dimensional manifolds, the Hilbert space is traditionally constructed as a space of integrable functions with respect to a measure on the manifold. Constructing the right measure is difficult since there is no translation invariant measure that could take the place of Lebesgue measure in finite dimensions. Thus geometric quantization becomes an ad hoc procedure in each particular case. On the other hand, the coherent space approach generalizes without severe problems to infinite dimensions. Second quantization thus appears as the theory of highest weight representations of infinite-dimensional Lie groups, or rather its coherent space version, which is somewhat simpler to manage. That it works in 2 dimensional spacetime is illustrated by the success of conformal field theory which has a rigorous mathematical description in terms of highest weight representations of the Virasoro group.

This interpretation of the quantum world in terms of the classical is important in quantum field theory when it comes to the explanation of perturbatively inaccessible phenomena such as particle states corresponding to solitons, or tunneling effects related to instantons. See JACKIW [25]; however, his explanations are mathematically vague. A coherent space setting makes this mathematically rigorous, at least in the semiclassical approximation.

Form factors. Form factors appear as coefficients of operators in the algebra of quadratics in the defining fields satisfying a conservation law (i.e., vanishing divergence). They determine the possible interactions with gauge fields. A good description of form factors requires a detailed knowledge of the causal irreducible representations of the Poincare group. See, e.g., WEINBERG [55, 56], and KLINK [32]; the results there are not manifestly covariant. The coherent space approach can be used to give nicer, manifestly covariant formulas. The form factors of a theory are needed for a subsequent analysis of spectral properties, such as the Lamb shift in QED.

Causal coherent manifolds. A **spacetime** is a smooth real manifold M with a Lie group $\mathbb{G}(M)$ of distinguished diffeomorphisms called **spacetime symmetries** and a symmetric, irreflexive **causality relation** \times on M preserved by $\mathbb{G}(M)$. We say that two sections j, k of a vector bundle over M are **causally independent** and write this as $j \times k$ if

$$x \times y \quad \text{for } x \in \text{Supp } j, y \in \text{Supp } k.$$

Here $\text{Supp } j$ denotes the support of the function j .

A **causal coherent manifold** over a spacetime M is a coherent manifold Z with the following properties:

- (i) The points of Z form a vector space of smooth sections of a vector bundle over M .
- (ii) The symmetries in $\mathbb{G}(M)$ act as unitary coherent maps.
- (iii) The coherent product satisfies the following causality conditions:

$$K(j, j') = 1 \quad \text{if } j \times j' \text{ or } j \parallel j' \quad (33)$$

$$K(j + k, j' + k) = K(j, j') \quad \text{if } j \times k \times j'. \quad (34)$$

Examples of important spacetimes include:

- (i) **Minkowski spacetime** $M = \mathbb{R}^{1 \times d}$ with a Lorentzian inner product of signature $(+1, -d)$ and $x \times y$ iff $(x - y)^2 < 0$. Here d is the number of spatial dimensions; most often $d \in \{1, 3\}$. $\mathbb{G}(M)$ is the **Poincaré group** $ISO(1, d)$.
- (ii) **Euclidean spacetime** M with $x \times y$ iff $x \neq y$. Two Euclidean cases are of particular interest:
- (iii) For **Euclidean field theory**, $M = \mathbb{R}^4$ and $\mathbb{G}(M)$ is the group $ISO(4)$ of Euclidean motions.
- (iv) For **chiral conformal field theory**, M is the unit circle and $\mathbb{G}(M)$ is the **Virasoro group**. Its center acts trivially on M but not necessarily on bundles over M .

To give examples of a causal coherent manifold, we mention that from any Hermitian quantum field ϕ of a relativistic quantum field theory satisfying the **Wightman axioms**, for which the smeared fields $\phi(j)$ (with suitable smooth real test functions j) are self-adjoint operators, and any associated state $\langle \cdot \rangle$, the definition

$$K(j, j') := \langle e^{-i\phi(j)} e^{i\phi(j')} \rangle$$

defines a causal coherent manifold.

There are many known classes of relativistic quantum field theories satisfying these properties in 2 and 3 spacetime dimensions. Under additional conditions one can conversely derive from a causal coherent manifold the Wightman axioms for an associated quantum field theory. In 4 spacetime dimensions, only free and quasifree examples satisfying the Wightman axioms are known. The question of the existence of interacting relativistic quantum field theories in 4 spacetime dimensions is completely open.

Many tools from finite-dimensional analysis, in particular the Lebesgue integral, Liouville measure, and averaging over compact sets must be replaced by more unwieldy constructs, and limits need much more careful considerations. Lack of heeding this would lead to the familiar ultraviolet (UV) divergences and infrared (IR) divergences of conventional quantum field theories. The IR and UV divergences go away if the mathematically rigorous and correct considerations are applied. This can be seen in quantum field theories in 2 and 3 spacetime dimensions.

It is an open problem how to achieve the same in interacting quantum field theories in the most important case, 4-dimensional spacetime. There it is only known how to avoid the UV divergences, using careful distribution splitting techniques in the context of causal perturbation theory. However, this approach only gives constructions for asymptotic series and misses the nonperturbative contributions needed for a fully defined interacting relativistic quantum field theories in 4 spacetime dimensions.

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