Black-box optimization benchmarking of the GLOBAL method

László Pál pallaszlo@sapientia.siculorum.ro
Faculty of Economic and Human Sciences, Sapientia – Hungarian University of Transylvania, Miercurea-Ciuc, Romania

Tibor Csendes csendes@inf.u-szeged.hu
Institute of Informatics, University of Szeged, Hungary

Mihály Csaba Markót Mihaly.Markot@univie.ac.at
Faculty of Mathematics, University of Vienna, Austria

Arnold Neumaier Arnold.Neumaier@univie.ac.at
Faculty of Mathematics, University of Vienna, Austria

Abstract

GLOBAL is a multistart type stochastic method for bound constrained global optimization problems. Its goal is to find the best local minima that are potentially global. For this reason it involves a combination of sampling, clustering, and local search. The role of clustering is to reduce the number of local searches by forming groups of points around the local minimizers from a uniformly sampled domain and to start few local searches in each of those groups. We evaluate the performance of the GLOBAL algorithm on the BBOB 2009 noiseless testbed, containing problems which reflect the typical difficulties arising in real-word applications. The obtained results are also compared with those obtained form the simple multistart procedure in order to analyze the effects of the applied clustering rule. An improved parametrization is introduced in the GLOBAL method and the performance of the new procedure is compared with the performance of the MATLAB GlobalSearch solver by using the BBOB 2010 test environment.

Keywords

Global optimization, stochastic search, clustering, multistart method, benchmarking.

1 Introduction

In this paper, global optimization problems subject to variable bound constraints are considered:

$$\min_{x \in X} f(x), \quad X \subset \mathbb{R}^D, \tag{1}$$

where $f(x)$ is the objective function, X is the set of feasibility, a rectangular domain defined by bounds on the variables and D is the dimension of the search space. In general we assume that the objective function is twice continuously differentiable, although this is not necessary for the global optimization framework procedure – with a proper local search algorithm also nondifferentiable problems can be solved.

Several stochastic strategies have been developed recently in the past in order to solve problem (1). Usually they consist of two phases: the global and the local one. During the global phase, random points are drawn from the search space X according to a certain, often uniform, distribution. Then, the objective function is evaluated in these points. During the local phase the sample points are manipulated by means of local search to yield a candidate global minimum. We assume that a proper local search method LS is available. It can be started from an arbitrary point $x_0 \in X$ and then this algorithm generates the sequence of points in X which converges to some $x^* := LS(x_0) \in X$, that is the local minimizer related to the starting point x_0.

These methods are also called *Multistart* techniques, because they apply local searches to each point in a random sample drawn from the feasible region (Boender et al., 1982b; Rinnooy Kan and Timmer, 1987a,b). However, the Multistart method is inefficient when it performs local searches starting from all sample points. That is, in such cases some local minimizer points will be found several times. Since local search is the most time consuming part of the method, it should ideally invoked no more than once in every *region of attraction*.

Various improvements were proposed by diverse authors in order to reduce the number of local searches, see e.g. (Törn, 1978; Rinnooy Kan and Timmer, 1987a; Guss et al., 1995). The two most important methods which are aimed at reducing the number of performed local searches are: the *clustering* methods and the *Multi Level Single Linkage (MLSL)* algorithms.

The basic idea behind clustering methods is to form groups (clusters) around the local minimizers from a uniformly sampled domain and to start as low number of local searches as possible in each of those groups. In other words, the procedure tries to identify the regions of attraction of the given function.

MLSL methods have been derived from clustering methods (Rinnooy Kan and Timmer, 1987b). In this algorithm the local search procedure is applied to every sample point, except if there is another sample point within some critical distance which has a lower function value.

Random Linkage (RL) multistart algorithms introduced by Locatelli and Schoen (Locatelli and Schoen, 1999) retain the good convergence properties of MLSL. Uniformly distributed points are generated one by one, and LS is started from each point with a probability given by a nondecreasing function $\phi(d)$, where d is the distance from the current sample point to the closest of the previous sample points with a better function value.

The multistart clustering global optimization method called GLOBAL (Csendes, 1988) has been introduced in the 80s for bound constrained global optimization problems with black-box type objective functions. The algorithm is based on Boender's algorithm (Boender et al., 1982b), and its goal is to find the best local minimizer points that are potentially global. The local search procedure used by GLOBAL was originally either a quasi-Newton procedure with the Davidon–Fletcher–Powell (DFP) update formula (Davidon, 1959) or a random walk type direct search method called UNIRANDI (for details see (Järvi, 1973)). The main idea behind quasi-Newton methods is the construction of a sequence of matrices providing improved approximation of the Hessian matrix (or its inverse) by applying rank-one (or rank-two) update formula in order to avoid the direct and costly calculations. The DFP formula was the earliest scheme for constructing the inverse Hessian and it has theoretical properties that guarantee superlinear (fast) convergence rate and global convergence under certain conditions. GLOBAL was originally coded in the Fortran and C languages. In several recent comparative studies (e.g. (Mongeau et al., 2000), (Moles et al., 2003)), this method performed quite well in terms of both efficiency and robustness, obtaining the best results in many cases.

Based on the old GLOBAL method, we introduced a new version (Csendes et al., 2008) coded in MATLAB. The algorithm was carefully analyzed and it was modified in some places to achieve better reliability and efficiency while allowing higher dimensional problems to be solved. In the new version we use the quasi-Newton local search method with the Broyden–Fletcher–Goldfarb–Shanno (BFGS) update instead of the earlier DFP. Numerical experiments (Powell, 1986) have shown that the performance of the BFGS formula is superior over the DFP formula. All three versions (Fortran, C, MATLAB) of the algorithm are freely available for academic and nonprofit purposes at

`www.inf.u-szeged.hu/∼csendes/Reg/regform.php`

(after registration and limited for low dimensional problems).

The aim of the present work is to benchmark the GLOBAL algorithm and compare the performance of it with a simple multistart procedure and with the MATLAB GlobalSearch solver on a testbed which reflects the typical difficulties arising in real-word applications. In the first comparison the main goal was to examine the benefits of the clustering procedure over the simple multistart type method, while in the second case the comparison of the global phase of the two methods was our target. The remainder of the paper is organized as follows: The GLOBAL method is presented in Section 2 and the test environment in Section 3. The benchmarking on the BBOB 2009 noiseless testbed (Finck et al., 2009a; Hansen et al., 2009b) is done in Section 4 and it is based on the unpublished report Pál et al. (2009). During this section, we also describe the parameters and its settings in the test. The CPU timing experiment is presented in Section 4.2, while the discussion of the results is done in Section 4.3. The comparison of the GLOBAL and the simple multistart procedure is described in Section 4.4.

In Section 5, we compare GLOBAL with the MATLAB GlobalSearch method using the BBOB 2010 test environment. The new parameter settings of the GLOBAL method are presented in Section 5.1, while the GlobalSearch method is overviewed in Section 5.2. Section 5.3 contains the comparison results.

2 Presentation of the GLOBAL algorithm

The GLOBAL method has two phases: a global and a local one. The global phase consists of sampling and clustering, while the local phase is based on local searches. The local minimizer points are found by means of a local search procedure, starting from appropriately chosen points from the sample drawn uniformly within the set of feasibility. In an effort to identify the region of attraction of a local minimizer, the procedure invokes a clustering algorithm. The role of clustering is to reduce the number of local searches by forming groups of points around the local minimizers from a uniformly sampled domain and start local searches as few times as possible in each of those groups. Clusters are formed stepwise, starting from a *seed point*, which may be an unclustered point with the lowest function value or the local minimum found by applying local search to this point. New points are attached to the cluster according to clustering rules.

GLOBAL uses the Single Linkage clustering rule (Boender et al., 1982b; Rinnooy Kan and Timmer, 1987a), which is constructed in such a way that the probability that a local method will not be applied to a point that would lead to an undiscovered local minimizer diminishes to zero when the size of the sample grows. In this method the clusters are formed sequentially and each of them is initiated by a seed point. The distance between two points x and x' in the neighborhood of the seed point x_s is defined as

$$d(x, x') = ((x - x')^\top H(x_s)(x - x'))^{1/2},$$

Algorithm 1: The GLOBAL algorithm

1. **function** GLOBAL(f, X)
2. $k \leftarrow 0; X^* \leftarrow \emptyset; X^{(1)} \leftarrow \emptyset$
3. **repeat**
4. $k \leftarrow k + 1$
5. Generate N points $x_{(k-1)N+1}, \ldots, x_{kN}$ with uniform distribution on X.
6. Determine the reduced sample consisting of the γkN best points from the cumulated sample (entire history) $x_1, \ldots, x_{kN}$.
7. Apply clustering to the reduced sample using the points of X^* and $X^{(1)}$ as seed points.
8. **while** *Not all points from the reduced sample have been clustered* **do**
9. Let $\overline{x}$ be the best unclustered point from the reduced sample.
10. $x^* \leftarrow LS(\overline{x})$
11. $C(x^*) \leftarrow C(x^*) \cup \{\overline{x}\}$
12. **if** $x^* \notin X^*$ **then**
13. $X^* \leftarrow X^* \cup \{x^*\}$
14. $x_s \leftarrow x^*$
15. **else**
16. $X^{(1)} \leftarrow X^{(1)} \cup \{\overline{x}\}$
17. $x_s \leftarrow \overline{x}$
18. **end**
19. Apply clustering to the unclustered points using x_s as seed point.
20. **end**
21. **until** *Some global stopping rule is satisfied.*
22. **return** *The smallest local minimum value found.*

where $H(x_s)$ is the Hessian of the objective function at the seed point. If x_s is a local minimizer then a good approximation of $H(x_s)$ can be obtained by using the BFGS method, otherwise $H(x_s)$ can be replaced by the identity matrix. Let $C(x_s)$ denote the cluster initiated by the seed point x_s. After a cluster $C(x_s)$ is initiated, we find an unclustered sample point x such that $d(x, y)$ is minimal, where $y \in C(x_s)$. This point is then added to $C(x_s)$, after which the procedure is repeated until $\min_{y \in C(x_s)} d(x, y)$ exceeds some critical value r_k. The applied critical distance in our algorithm is based on the one used in (Boender et al., 1982a) which is

$$r_k = \frac{1}{\sqrt{\pi}} \left(\Gamma(1 + \frac{D}{2}) \cdot |H(x_s)|^{1/2} \cdot m(X) \cdot (1 - \alpha^{1/(kN-1)}) \right)^{1/D},$$

where Γ is the gamma function, $|H(x_s)|$ denotes the determinant of $H(x_s)$, $m(X)$ is the Lebesgue measure of the set X, and $\alpha \in (0, 1)$ is a parameter of the clustering procedure. The main steps of GLOBAL are summarized in Algorithm 1.

In line 2, the X^* and $X^{(1)}$ sets are initialized, where X^* is a set containing the local minimizer points found so far, while $X^{(1)}$ is a set containing sample points to which the local search procedure has been applied unsuccessfully in the sense that already know local minimizer was found again. Moreover, the set $X^{(1)}$ has the role to further reduce the number of local searches by applying clustering using the elements of it as seed points. The number of new drawings is denoted by k, set initially to 0. The algorithm contains a main iteration loop, steps from line 4 to line 20, that will be repeated until some global stopping rule is satisfied. In line 5, N points are generated uniformly on X. In line 6, a reduced sample is constructed by taking those $\gamma k N$ points of the cumulated sample that have the lowest function values. The cumulated sample contains all points sampled during the iterations. A clustering procedure is then applied to the reduced sample (line 7). The elements of X^* are first chosen as seed points, followed by the elements of $X^{(1)}$. In case of a seed point x_s, we add all unclustered reduced sample points which are within the critical distance r_k to the cluster initiated by x_s. In the first iteration, X^* and $X^{(1)}$ are empty and thus no clustering takes place.

Between lines 8 and 20 we iterate over the unclustered points from the reduced sample and apply a local search procedure to them to find a local minimizer point x^*. The point $\overline{x}$ is then added to the $C(x^*)$ cluster (line 11). If x^* is a new local minimizer point, then we add it to X^* (line 13) and choose it as the next seed point (line 14), otherwise we add $\overline{x}$ to $X^{(1)}$ (line 16) and choose it as the next seed point (line 17). In line 19, we apply again the clustering procedure to the unclustered reduced sample points which are within a critical distance from the cluster initiated by the seed point x_s. In line 22, the smallest local minimum value is returned.

One of the questions in applying a stochastic method is when to stop it. Several approaches based on different assumptions about the properties of possible objective functions f and using some stochastic techniques have been proposed to design a proper stopping rule.

A Bayesian stopping rule for the Multistart algorithm has been introduced by (Zieliński, 1981) and further developed by (Boender and Zieliński, 1982; Boender and Rinnooy Kan, 1987, 1991; Betrò and Schoen, 1992) and others.

Most Bayesian stopping rules for multistart techniques are based on the collected knowledge about the size of the sample and the number of local minimizers detected. In our GLOBAL algorithm we stop the search (line 21) when it has not found any new local minimizer point in the actual iteration step. Beside this principal stopping criteria, GLOBAL contains further stopping rules in order to stop the optimization process when this takes too long. The first one stops the algorithm when it exceeds the upper limit on the number of iterations, while the second one stops the search when the number of the found local minimizer points is larger than a prescribed value.

3 The test environment description

In this paper, the numerical experiments are conducted on a testbed comprising 24 noiseless test functions (Finck et al., 2009a; Hansen et al., 2009b). These functions have been constructed so that they reflect the real-word application difficulties and are split into several groups like separable functions ($f_1 - f_5$), functions with low or moderate conditioning ($f_6 - f_9$), functions with bad conditioning and unimodal ($f_{10} - f_{14}$), multi-modal with adequate global structure ($f_{15} - f_{19}$), multi-modal with weak global structure ($f_{20} - f_{24}$). All functions are scalable with the dimension, thus in our tests we used 2, 3, 5, 10 and 20 as dimensions. Additionally, all functions are defined over $\mathbb{R}^D$, while the actual search domain is $[-5; 5]^D$. Every function has an artificially chosen optimal function value. Consequently, for each function different instances can be generated. Each function is tested over five different instances and the experiments are repeated three times for each instance. The performance of the algorithm is evaluated over all 15 trials. The success criterion of a run is to reach the $f_t = f_{opt} + \Delta f_{\mathrm{t}}$ target value, where f_{opt} is the (pre-known) optimal function value, and Δf_{t} is the precision to reach.

In order to quantify the search cost of an algorithm, a performance measure should be provided. The main performance measure adopted in this paper (Hansen et al., 2009a; Price, 1997) is the runtime ERT, Expected Running Time. The ERT number depends on a given target function value, and is computed over all relevant trials as the number of function evaluations used during the trials until the best function value did not reach f_{t}, summed over all trials and divided by the number of trials that actually reached f_{t}. Formally

$$ERT(f_{\mathrm{t}}) = \frac{p_S \cdot RT_S + (1 - p_S) \cdot RT_{US}}{p_S},$$

where p_S is the probability of success, the ratio of the number of successful runs over the total number of runs, RT_S and RT_{US} denote the average number of function evaluations for successful and unsuccessful trials, respectively.

The results are also presented using the Empirical Cumulative Distribution Function (ECDF) of the distribution of ERT divided by D to reach a given target function value. This shows the empirical cumulated probability of success on the problems considered depending on the allocated budget. For a more detailed environment and experimental description see (Hansen et al., 2009a, 2010a).

4 Benchmarking GLOBAL on the BBOB 2009 noiseless testbed

4.1 Parameter tuning and setup

GLOBAL has six parameters to set: the number of sample points to be generated within an iteration step, the number of best points to be selected for the reduced sample, the stopping criterion parameter for the local search, the maximum number of function evaluations allowed for local search, the maximum number of local minima to be find, and the type of local method to be used. All these parameters have a default value and usually it is enough to change only the first three of them.

In all dimensions and for all functions we used 300 sample points, and the two best points were kept for the reduced sample. In 2, 3, and 5 dimensions we used the Nelder-Mead simplex method (Nelder and Mead, 1965) implemented by (Kelley, 1999) as a local search with 10^{-8} as termination tolerance parameter value and with 5000 as the maximum number of function evaluations. In 10 and 20 dimensions with the f_3, f_4, f_7, f_{16}, f_{23} functions we used the previous settings with a local search tolerance of 10^{-9}. Finally, in the case of the remaining functions we used the MATLAB's `fminunc` function as the local search method using the BFGS update formula with 10000 as the maximum number of function evaluations and with 10^{-9} as the termination tolerance parameter value.

As it can be observed, during the parameter tuning we used two different settings. In lower dimensions we used the Nelder-Mead method while in higher dimensions the BFGS local search was applied to all functions except for five of them. Although this kind of a priori parameter settings are not suggested in general, the two important parameters of GLOBAL (the number of sample points, the number of best points selected) were the same on the entire testbed. The different settings may be characterized with the entropy measure crafting effort (Price, 1997; Hoos and Stützle, 1998) for each dimensionality in the following way:

$$\mathrm{CrE} = -\sum_{k=1}^{K} \frac{n_k}{n} \ln\left(\frac{n_k}{n}\right),$$

where $n = \sum_{k=1}^{K} n_k$ is the number of functions in the testbed and n_k is the number of functions, where the parameter setting with index k was used for $k = 1, \ldots, K$, K is the number of different parameter settings. The crafting effort $\mathrm{CrE} = 0$ for dimensions 2, 3, and 5, while for $D = 10, 20$ it can be calculated as $\mathrm{CrE}_{10} = \mathrm{CrE}_{20} = -(\frac{5}{24} \ln \frac{5}{24} + \frac{19}{24} \ln \frac{19}{24}) = 0.5117$.

4.2 CPU timing experiment

For the timing experiment the GLOBAL algorithm was run on the test function f_8, and restarted until at least 30 seconds had passed (according to Figure 2 in (Hansen et al., 2009a)). These experiments have been conducted with an Intel Core 2 Duo 2.00 GHz processor computer under Windows XP using the MATLAB 7.6.0.324 version. We have completed two experiments using the BFGS and the simplex local search methods. The other algorithm parameters were the same. In the first case (BFGS) the results were $(2.8, 2.9, 3.0, 3.0, 3.2, 3.2) \cdot 10^{-4}$ seconds, while in the second case (Nelder-Mead simplex) they were $(2.6, 2.9, 3.4, 4.6, 7.5, 21.0) \cdot 10^{-4}$ seconds per function evaluation in dimensions 2, 3, 5, 10, 20, and 40, respectively. The CPU time of a function evaluation of the BFGS search grows sub-linearly with the dimension. The slow increase in the CPU time is due to the initializing process. On the other hand in lower dimensions there will be more restarts (before surpassing the 30 seconds) which means that there will be more initializations. We assume the CPU time per function evaluation would increase given that the dimensionality is large enough. For the Nelder-Mead simplex method, the CPU time increases with the dimension linearly up to 20 dimensional problems, while for 40 dimensional functions a rapid increase can be observed.

4.3 Results and discussion

Results from experiments according to (Hansen et al., 2009a) on the benchmark functions given in (Finck et al., 2009a; Hansen et al., 2009b) are presented in Figure 1 and Tables 2 and 3.

Tables 2 and 3 give the Expected Running Time (ERT) for target values $10^{1,\,0,\,-1,\,-3,\,-5,\,-7}$ divided by the best ERT obtained during BBOB 2009, together with its standard deviation (smaller values in parentheses), for $D = 5$ and $D = 20$. The median number of conducted function evaluations is additionally given in *italics*, if $\mathrm{ERT}(10^{-7}) = \infty$. #succ is the number of trials that reached the final target $f_{\mathrm{opt}} + 10^{-8}$. The ERT values can also be followed in the Figure 1. Numbers above ERT-symbols indicate the number of successful trials. The thick line with diamonds shows the single best results from BBOB 2009 for $\Delta f = 10^{-8}$. Additional grid lines show linear and quadratic scaling.

For low search space dimensions the algorithm shows good results on many functions. The number of solved functions amounts to 18, 16, 11, 8, 5 out of 24 functions for dimensions 2, 3, 5, 10, 20. We can notice that GLOBAL obtains the highest number of successful trials in separable, moderate, ill-conditioned and weak structure noiseless functions, specifically for f_1, f_2, f_5, f_6, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{21} and f_{22} in dimensions 2, 3, and 5. For f_1, f_2, f_5, f_8 and f_9, the method obtained successful trials for all dimensions.

The scaling of the expected number of function evaluations with the problem dimension is closely linear for f_8, f_9 and is approximately quadratic for f_2. For f_1 and f_5 we can observe a decreasing tendency (see Figure 1). These results are due to the stochastic nature of the method, usually the ERT grows sub-linearly on these functions. The running times to reach the final target function value in case of the solved problems in 20 dimension range between $25D$ and $2500D$.

Considering the different function subgroups, the best behavior of the GLOBAL algorithm can be observed on the separable (except f_3 and f_4), moderate (except f_7) and ill-conditioned functions. The good performance on these subgroups are due to the unimodal property of the objective functions. On the other hand, most of these functions have a quite large region of attraction of the global optimum (e.g. f_8 and f_9), hence there is a high chance to sample in this ones. In case of the f_6 (Attractive sector) and f_{11} (Discus) functions in dimension 20 up to the target precision 10^{-1}, all the 15 trials were successful, but the method fails to reach $\Delta f = 10^{-3}$. The results on the attractive sector function can be improved by increasing the function evaluation limit of the BFGS method, while for the Discus function f_{11} one cannot find a better target precision value than 10^{-1} in 20-D due to a problem of the BFGS local search. In this case the local search method stops too early because it cannot decrease the objective function along the current search direction. Finding the final target function values for f_8 and f_9 are mainly due to the BFGS local search and partly to the property of these functions presented previously. GLOBAL performs also well on Gallagher's multimodal functions f_{21} and f_{22} with weak global structure. Compared to the best algorithm from BBOB 2009, the GLOBAL method can improve the ERT in dimensions 5 and 20 on the latter functions.

The hardest problems for which the method did not reach the solution in higher dimensions are the multimodal Rastrigin functions f_3, f_4, f_{15}, f_{24}. In case of the last one even in 2-D we could not find a better target precision value than 10^{-2}, while in the case of the functions f_3, f_4, and f_{15} the Δf_{best} value is not better than 10 in 5-D and 10^2 in 20-D, respectively. The common feature of these functions is that they have more than 10^D local optima. Therefore the algorithm cannot discover the overall function structure. Moreover, the size of the basin of attraction of the global optimum is small for these problems, hence the algorithm fails to satisfactorily sample in these regions. GLOBAL also fails to reach a target value below 1 on the multimodal functions f_{17} and f_{19} with adequate global structure in 5-D and 10 in 20-D. The reason of this is the same as presented above.

Considering the individual maximum number of function evaluations, GLOBAL performs well on ill-conditioned functions and on the multimodal weakly structured functions for a budget smaller than a thousand times D. A similar conclusion has been reached in (Hansen et al., 2010b), where 31 algorithms were compared on a testbed of functions in dimensions up to 40. GLOBAL was ranked together with NEWUOA (Powell, 2006) and MCS (Huyer and Neumaier, 1999) best for a function evaluation budget of up to $500D$ function values, but was no longer competitive when the budget was significantly larger.

4.4 Clustering effect vs. simple multistart

As we described earlier the clustering is applied to decrease the number of local searches. In this subsection we investigate the effect of the clustering procedure on the efficiency of the GLOBAL algorithm and we compare it with a simple multistart (MSTART) procedure where 10^4 independent local searches are started from randomly generated points. In other words, we restart the search whenever the local search has converged to a local optimum until the maximum number of function evaluations is reached. In this experiment, we used 10^5 as the maximum number of function evaluations. These parameter settings are reasonable since for the GLOBAL method we used similar values. Regarding the local search and its parameters, we use the same scheme (depending on the function and dimension) as in the case of GLOBAL. The comparisons are based on the GLOBAL results obtained during BBOB 2009 (Tables 2 and 3) and on the results of the simple multistart procedure contained by the same tables. The consequences of clustering can nicely be followed in the Figures 2 and 3, where the results for different subgroups are aggregated in the ECDF graphs of ERT for 5-D and 20-D functions. Like in the Figures 2 and 3, in our comparisons we use the $\log_{10}$ of the ERT/D as a measure for the budget of function evaluations.

Two important aspects can be observed from the Figures 2 and 3. The first one is that for low budgets of function evaluations (smaller than 2) the simple multistart procedure is faster than GLOBAL. This is due to the fact that GLOBAL starts the first local search after the evaluation of the randomly generated 300 sample points. The second important aspect is that GLOBAL stops the search after the budget of 4 evaluations. This early stopping is caused on one hand by the limit of the number of iterations and on the other hand by the limit number of the found new local minimizers. In the first case GLOBAL stops if the cumulated number of best points selected during all iterations are greater than 100, while in the second case it stops when the number of the found local minimizers are greater than 20. Based on the two important observations the impact of clustering can be analyzed between budgets of 2 and 4 evaluations.

In the 5-D space, we can observe that after the first iteration of GLOBAL (after the evaluation of the 300 sample points) the proportion of solved problems suddenly begins to grow. For larger budgets than 2 on separable and moderate functions, GLOBAL takes over the simple multistart procedure by solving a slightly larger number of problems. On the ill-conditioned function group the simple multistart is slightly faster than GLOBAL. This increase is caused by solving one instance of the f_{14} function by the simple multistart method. The differences between the two methods are nuanced on these subgroups since most of the functions are unimodal. On some functions (f_7, f_{11}, f_{12}) the simple multistart method is also more reliable (it solves more instances) than GLOBAL. GLOBAL stops too early on these functions caused on one hand by the main stopping criterion (on f_7 and f_{12}) and on the other hand by exceeding the upper limit of found local minimizers (on f_{11}). The impact of the clustering rule is more pronounced on multi-modal and multi-modal with weak structure functions. On this function groups GLOBAL clearly outperforms the simple multistart procedure on the middle stage of the optimization process.

In the 20-D space, we can notice similar behavior as in 5-D of the two methods. The disadvantage of GLOBAL in the beginning of the optimization process is negligible compared with the simple multistart in the case of moderate, ill-conditioned and multi-modal function groups. This is due to the difficulties related to higher dimensions. On all function groups GLOBAL is slightly faster than the simple multistart for larger budgets than 2 evaluations. The increase is significantly larger on ill-conditioned and multi-modal with weak structure functions.

All in all, the impact of the clustering rule can be recognized on almost every function group, consequently, GLOBAL is faster than the simple multistart procedure on the middle stage of the optimization. Naturally, the differences are more pronounced on the multimodal functions. Furthermore, in the next section we propose some improvements in order to reduce the two drawbacks of the GLOBAL method: the slowness in the initial stage of the optimization and the too early stopping of the algorithm.

5 Comparing GLOBAL with the MATLAB GlobalSearch solver using the BBOB 2010 framework

5.1 Improved parameter settings in GLOBAL

The most important parameters used in GLOBAL are the number of sample points to be drawn uniformly in one iteration cycle, the number of best points selected from the actual sample, and the maximal number of function evaluations allowed for a local search. In our recent work (Csendes et al., 2008), we tuned this parameters empirically and individually for a set of problems without directly including information like the dimension of the problem or the maximal function evaluation budget.

Although, GLOBAL has its own stopping criteria, we introduced a new parameter `maxfunevals` which controls the total number of function evaluations. This parameter is also used in setting the default value of the sample number and the maximal number of function evaluations for local search in the following way:

$$\text{number\_of\_sample\_points} = \min(50 * D, maxfunevals * 1\%),$$

$$\text{func\_eval\_nr\_in\_local\_search} = maxfunevals * 10\%.$$

As a consequence of the new settings the sample size increases by the dimension which is more flexible than the previous settings (fixed sample size in all dimensions). In our tests, we set the function evaluation budget to $maxfunevals = 2 * 10^4 * D$, hence the upper limit of the function evaluations in 20 dimension is $4 * 10^5$. The number of the best points to be clustered was set to D. In the new settings, we eliminated all the stopping conditions of the GLOBAL in order to avoid the too early stopping of the method. Thus, the only stopping rule applied is the upper limit of the number of function evaluations. It can be observed that the limit of function evaluation budget is set twice larger than the limit obtained ($10^4 D$) in the pervious experiments in order to give a greater chance for the GLOBAL method to solve a problem.

Another important setting in the GLOBAL method is to start a single local search method before the first sampling. The aim of this modification is to speed-up the algorithm in the initial stage of the optimization. On the whole testbed we use the MATLAB's `fmincon` local search method in all dimensions. `fmincon` is a gradient based interior-point algorithm that aims to find the local minimum of a constrained nonlinear multivariable function. Nevertheless, it can be applied to bound constrained problems too. The termination tolerance parameter value of the local search method was set to 10^{-12}. In this experiment we do not further use the simplex method since the only available local search method for the GlobalSearch solver is the `fmincon`. Hence we have the possibility to compare the global phases of the two methods. As we use the same settings for all functions in all dimensions, the corresponding crafting-effort is equal to $\mathrm{CrE} = 0$.

5.2 Overview of the MATLAB GlobalSearch solver

GlobalSearch is a solver designed to find global optima of smooth constrained nonlinear problems. The solver was introduced in the MATLAB R2010a version and is available in the new Global Optimization Toolbox. It is a multistart type method which runs a local search from a variety of starting points in order to find a global minimum, or multiple local minima. The solver uses a scatter-search mechanism for generating start points. It analyzes them and rejects those that are unlikely to improve the best local minimum found so far. Essentially, GlobalSearch accepts a start point only when

Algorithm 2: The GlobalSearch solver steps

1. **function** GlobalSearch(f, X, x_0)
2. Run `fmincon` from x_0.
3. Generate N_1 trial points using the scatter-search mechanism on X.
4. Stage 1: Start a local search from the best trial point among the first N_2 points.
5. Initialize the regions of attraction, counters, threshold, based on the point find in Stage 1 and in the first step of the algorithm.
6. **repeat**
7. Stage 2: Examine the a remaining trial points and run a local search from a point x, if x is not in any existing basin and $f(x) < threshold$.
8. **until** *Reaching MaxTime seconds or running out of trial points*
9. **return** *The smallest local minimum value found.*

it determines that the point has a good chance of obtaining a global minimum. The method uses MATLAB's `fmincon` function as the local search method.

The GlobalSearch solver is similar to the commercial optimization software TOMLAB/OQNLP; both of them are based on the paper (Ugray et al., 2007). Although GlobalSolver is not considered a state of the art algorithm, based on some recent comparative studies (Rios and Sahinidis, 2010) on bound constrained problems, its commercial counterpart show superior performance in terms of refining a near-optimal solution. The main reason of choosing the GlobalSolver for comparisons was that this method is very similar to the GLOBAL procedure, involving cluster formation by identifying the regions of attraction and avoiding local searches from every trial point. On the other hand, as we used GLOBAL with `fmincon`, our aim was to compare the two methods based on the different global phases used by them. The main steps of the GlobalSearch method are summarized in Algorithm 2.

The most important parameters of the GlobalSearch solver are the number of trial points (`NumTrialPoints`) with a default value of 1000, and the number of points used in Stage 1 (`NumStageOnePoints`) with a default value of 200. We do not have any possibility to control the total function evaluation budget, but we can impose a running time limit using the `MaxTime` parameter. In the conducted experiments we used $500D$ as the number of trial points while the number of points from the first stage was set to $100D$. It can be observed that number of trial points ($500D$) are larger than in the case of GLOBAL ($50D$). This is due to the fact that the GlobalSearch solver has only one main iteration, while GLOBAL has many. On the other hand the set value is enough to understand the main characteristics of the GlobalSearch method. The running time limit on a function instance was set to 120 seconds. As a result we obtained the same upper limit on the number of function evaluations as in the case of the GLOBAL method. The starting point (used in line 2, Algorithm 2) chosen by us is the center of the search domain $[-5; 5]^D$, but it can also randomly be selected.

5.3 Results

In this section we show the comparison results obtained for the GLOBAL and GlobalSearch algorithms. The conducted experiment results according to (Hansen et al., 2010a) on the benchmark functions given in (Finck et al., 2009b; Hansen et al., 2009c) are presented in the Tables 4 and 5 and Figures 4, 5, 6 and 7. ERT loss ratio versus given budget FEvals divided by dimension in log-log display in 5 and 20-D are presented in the Figures 8, 9, 10, and 11.

In Tables 4 and 5 we can follow the running time in terms of the number of function evaluations for dimension 5 and 20 in comparison with the respective best algorithm of BBOB 2009. These tables also present the **statistical significance** of the difference between the two methods, which is tested with the rank-sum test for a given target Δf_{t} using, for each trial, either the number of needed function evaluations to reach Δf_{t}, or, if the target was not reached, the best Δf-value achieved, measured only up to the smallest number of overall function evaluations for any unsuccessful trial under consideration if available. Bold entries are statistically significantly better (according to the rank-sum test) compared to the other algorithm, with $p = 0.05$ or $p = 10^{-k}$ where $k > 1$ is the number following the $\star$ symbol, with Bonferroni correction of 48.

Both algorithms perform very similar considering the number of solved problems (with precision $\Delta f_{\mathrm{t}} = 10^{-8}$) in 5 and 20-D. The GlobalSearch method solves 11 and 8 out of 24 functions in 5 and 20-D, while GLOBAL solves the same problems (except f_{19} in 5-D), and, in addition, f_{20} in 5-D and f_6 in 20-D.

Considering the ERT values we can observe that GLOBAL is faster than the GlobalSearch method almost on all functions. Moreover these results appear statistically significant on the f_1, f_5, f_9, f_{10}, f_{11}, f_{13}, f_{14} and f_{21} functions in 5 and 20-D. The most surprising results can be observed on the multi-modal functions f_{20}, and f_{21}, where GLOBAL outperforms the GlobalSearch method up to a factor of 72. Compared to the best algorithm from BBOB 2009, the GLOBAL method can improve the ERT in dimensions 5 and 20 on f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, and f_{21}. These improvements can be observed in the case of a few target values and usually are statistically significant. On the f_9 and f_{21} functions GLOBAL is obviously the best algorithm from BBOB 2009. The f_9 function is unimodal and continuous, hence it is more accessible for the fast `fmincon` local search method. The f_{21} method is multi-modal with moderate conditioning with relatively large regions of attraction. That is why the used clustering procedure of GLOBAL can approximate the level sets of the given function accurately.

Considering the number of solved instances up to the final precision we can note that usually GLOBAL is better than the GlobalSearch method. Significant differences can be observed on the f_{12}, f_{20}, f_{21}, and f_{22} functions in 5-D and on the f_6, f_{21}, and f_{22} functions in 20-D. GLOBAL solves at least twice as much instances on the latter functions than the GlobalSearch method. The largest difference is in the case of the f_{20} and f_6 functions in 5-D and 20-D, respectively. The GlobalSearch method could not solve any instances of these functions.

Figures 4, 5, 6 and 7 show the empirical cumulative distributions of the runtime in number of function evaluations and of the runtime ratio between the two algorithms in dimensions 5 and 10. The x-values in the figures show a given budget (a given number of function evaluations, divided by dimension), while the y-value gives the proportion of problems where the Δf_t-value was reached within the given budget.

The first observation is that GLOBAL is as fast on almost all function groups in the initial stage of the optimization as the GlobalSearch method. This is due to the included initial local search method. A significant difference can be read from the figures of the multimodal functions with weak structure, where GLOBAL clearly outperforms the GlobalSearch method. These results are due to the f_{21} and f_{22} functions which main characteristics are that they posses many local optima and the conditioning around the global optimum is about 30 and 1000, respectively. The GLOBAL algorithm uses the Single Linkage method which approximates the level sets more accurately on these functions, while GlobalSearch makes the assumption that basins of attraction are spher-

Algorithm	2-D	3-D	5-D	10-D	20-D	40-D	80-D
GLOBAL	6.8 (215)	6.1 (147)	5.3 (87)	4.3 (43)	3.8 (17)	3.5 (5)	3.6 (2)
GlobalSearch	9.8 (7)	9.5 (8)	7.9 (7)	5.8 (5)	4.6 (2)	3.8 (1)	4.1 (1)

Table 1: CPU time per function evaluation in seconds*10^{-4}
and the corresponding restarts numbers

ical which is not an ideal case.

In the middle- and final stage of the optimization GLOBAL provide similar or better results than the GlobalSearch method in 5 and 20-D. GLOBAL is much faster on the weak structure functions (in 5 and 20-D), due to the facts presented previously, and on the moderate functions (in 20-D), due to solving 14 instances of the function f_6.

Another important result of the improved settings of the GLOBAL method (by eliminating the original stopping rules) is that now the proportion of trials (see all functions groups) are growing continuously until the method reaches the maximum function evaluation number. This aspect can be observed particulary on the weak structure subgroup in 5-D and on the separable-, moderate-, and weak structure function groups in 20-D.

Considering the ERT loss ratios both algorithms show a very similar characteristic. For very small budgets (up to 10^2), the median (horizontal line) and geometric mean (connected line) of the ERT ratio are below ten in the case of the GLOBAL method (see Figure 8). The median values usually are larger in the case of the GlobalSearch solver (see Figure 10).

The timing experiment results for the two algorithms can be found in Table 1. It contains the CPU time per function evaluation and the corresponding number of restarts. As it can be observed, for both algorithms the necessary CPU time decreases with increasing dimension up to 40-D. In the case of the GLOBAL method, this is most likely due to a larger number of initialization procedures for the required multiple runs of the algorithm until thirty seconds have passed, while in the case of the GlobalSearch method there is no dependency to be recognized between the number of restarts and CPU time decrease. The relative small number of restarts are due to the lack of a proper termination criteria.

6 Summary and conclusions

We have benchmarked two different variants of the GLOBAL algorithm on a noiseless testbed using the BBOB 2009 and BBOB 2010 frameworks, respectively. As a result of the first experiment, we can state that the investigated method performs well on ill-conditioned and multimodal, weakly structured functions up to a small budget, but on multimodal functions the results are usually poorer. GLOBAL was compared with the simple multistart method in order to analyze the effects of the clustering procedure. The results clearly show that thanks to the clustering procedure GLOBAL is faster than the simple multistart procedure in the middle stage of the optimization.

In the second experiment we compared GLOBAL with an improved parametrization against the GlobalSearch solver. The improved parametrization consists of a flexible sampling phase where the number of sampled points and the number of selected best points depend on the dimension. An initial local search was also introduced, and all the stopping criterions used previously were eliminated. The only stopping rule applied was the maximum number of function evaluations. Based on the results obtained from the BBOB 2010 framework we can conclude that GLOBAL is similar to the GlobalSearch method in the initial stage of the optimization, while in the middle- and final stages of the process clearly outperforms the GlobalSearch procedure. Overall, with the improved GLOBAL method we succeeded to obtain better results in the initial and final stage of the optimization process. Further improvements may be possible in the final stage of the optimization but this assumes a more sophisticated sampling phase.

Acknowledgements. This work was supported by the grant TMOP-4.2.2/08/1/2008-0008 of the Hungarian National Development Agency. The authors would like to thank the anonymous reviewers for their useful constructive comments.

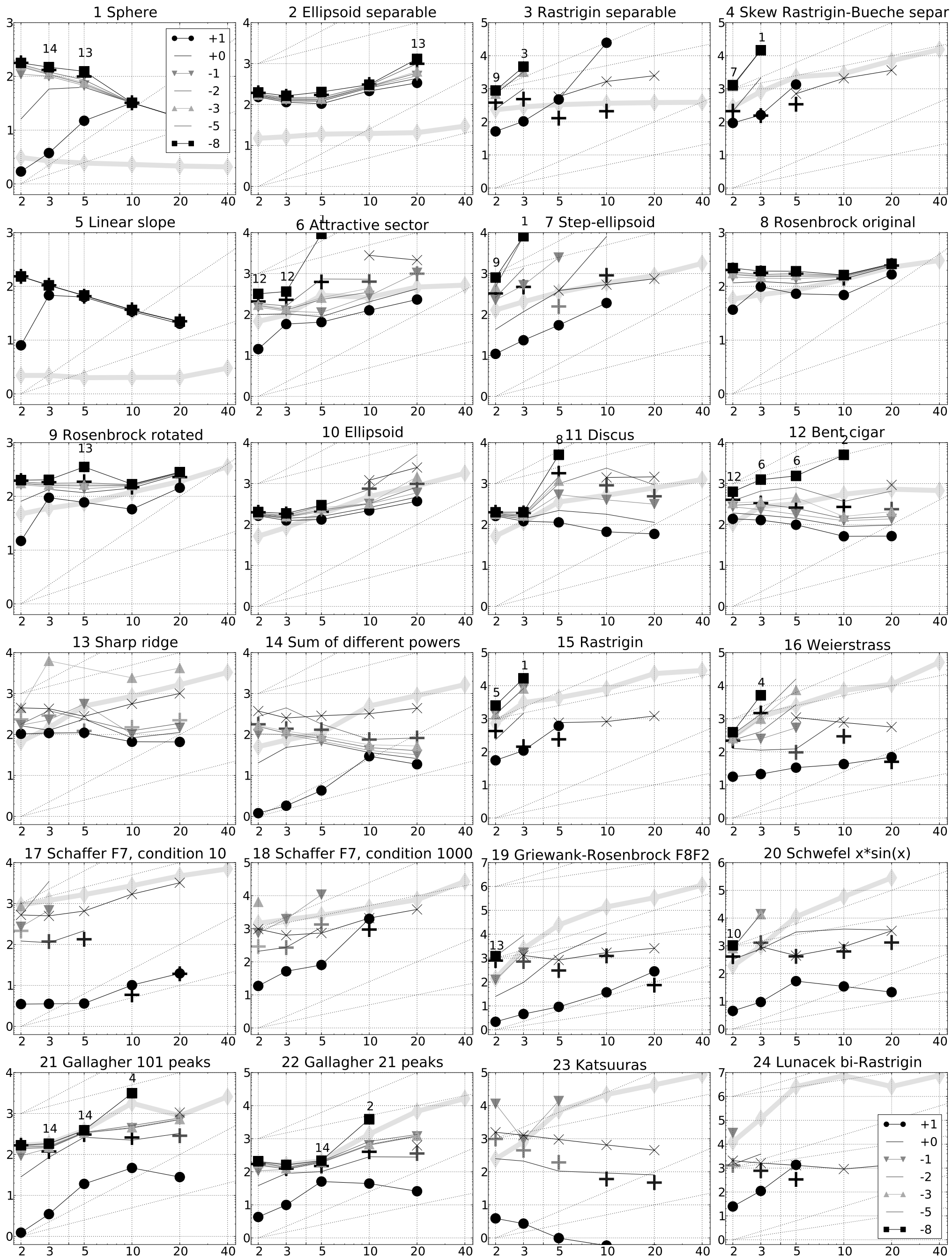


Figure 1: Expected running time (ERT) divided by dimension versus dimension in log-log presentation. Shown are different target values $f_{\rm opt} + \Delta f$, where $\Delta f = 10^{\{+1,0,-1,-2,-3,-5,-8\}}$ and the exponent is given in the legend of f_1 and f_{24}. Plus symbols (+) show the median number of f-evaluations for the best reached target value. Crosses (×) indicate the total number of f-evaluations (#FEs($-\infty$)) divided by the number of trials. Numbers above ERT-symbols indicate the number of successful trials. Y-axis annotations are decimal logarithms.

Δf	1e+1	1e+0	1e-1	1e-3	1e-5	1e-7	#succ
f_1	11	12	12	12	12	12	15/15
GLOBAL	6.8(9.4)	26(0.61)	28(0.74)	32(1.2)	35(1.0)	39(1.4)	13/15
MSTART	2.3(1.2)	4.7(1.1)	6.9(1.5)	11(1.3)	15(1.3)	19(1.8)	15/15
f_2	83	87	88	90	92	94	15/15
GLOBAL	6.3(1.9)	6.9(1.9)	7.3(1.8)	7.8(1.5)	8.2(1.4)	8.5(1.4)	15/15
MSTART	6.6(4.8)	8.3(3.1)	8.7(3.3)	9.2(3.8)	10(3.6)	10(3.6)	15/15
f_3	716	1622	1637	1646	1650	1654	15/15
GLOBAL	3.3(3.8)	∞	∞	∞	∞	∞*2600*	0/15
MSTART	4.6(5.2)	278(292)	∞	∞	∞	∞*1.0e5*	0/15
f_4	809	1633	1688	1817	1886	1903	15/15
GLOBAL	8.3(8.6)	∞	∞	∞	∞	∞*3200*	0/15
MSTART	11(12)	419(485)	∞	∞	∞	∞*1.0e5*	0/15
f_5	10	10	10	10	10	10	15/15
GLOBAL	32(1.3)	33(2.6)	34(2.4)	34(2.4)	34(2.4)	34(2.4)	15/15
MSTART	3.0(1.1)	4.1(1.2)	4.2(1.4)	4.2(1.4)	4.2(1.4)	4.2(1.4)	15/15
f_6	114	214	281	580	1038	1332	15/15
GLOBAL	2.9(0.21)	2.1(0.60)	2.0(0.50)	2.2(1.6)	3.6(3.5)	35(37)	1/15
MSTART	2.4(2.9)	2.7(2.1)	3.1(2.2)	4.3(5.8)	5.7(4.8)	27(41)	6/15
f_7	24	324	1171	1572	1572	1597	15/15
GLOBAL	12(6.7)	5.7(5.7)	10(12)	∞	∞	∞*1900*	0/15
MSTART	14(24)	10(13)	27(39)	155(172)	155(169)	152(169)	5/15
f_8	73	273	336	391	410	422	15/15
GLOBAL	5.0(0.34)	2.1(1.3)	2.1(1.1)	2.1(0.86)	2.1(0.86)	2.2(0.81)	15/15
MSTART	1.7(1.5)	2.4(2.0)	2.3(1.6)	2.3(1.4)	2.3(1.4)	2.3(1.4)	15/15
f_9	35	127	214	300	335	369	15/15
GLOBAL	11(2.2)	4.6(1.3)	3.2(0.80)	2.8(0.74)	2.7(0.60)	2.7(1.2)	13/15
MSTART	3.2(2.5)	3.9(2.9)	3.0(1.7)	2.5(1.2)	2.4(1.1)	2.3(1.0)	15/15
f_{10}	349	500	574	626	829	880	15/15
GLOBAL	1.9(0.70)	1.6(0.49)	1.8(0.71)	2.0(1.5)	1.7(1.1)	1.7(1.1)	15/15
MSTART	1.3(1.1)	1.3(0.77)	1.3(0.41)	1.3(0.40)	1.1(0.31)	1.1(0.28)	15/15
f_{11}	143	202	763	1177	1467	1673	15/15
GLOBAL	4.0(1.5)	5.5(2.6)	3.5(3.2)	5.0(7.8)	5.0(6.5)	8.5(8.3)	8/15
MSTART	5.3(3.6)	6.2(3.4)	2.9(2.1)	2.7(2.4)	2.3(2.2)	2.1(1.9)	15/15
f_{12}	108	268	371	461	1303	1494	15/15
GLOBAL	4.6(1.2)	2.7(0.61)	2.4(0.82)	5.0(6.7)	3.1(4.0)	3.4(4.0)	6/15
MSTART	4.2(4.1)	3.4(3.5)	3.2(3.1)	6.2(4.5)	5.2(7.7)	7.4(12)	11/15
f_{13}	132	195	250	1310	1752	2255	15/15
GLOBAL	4.2(2.5)	6.1(4.9)	11(11)	∞	∞	∞*1300*	0/15
MSTART	2.0(1.1)	2.8(3.4)	3.3(2.7)	75(80)	∞	∞*1.0e5*	0/15
f_{14}	10	41	58	139	251	476	15/15
GLOBAL	2.2(1.9)	7.7(0.22)	5.9(0.28)	3.3(0.40)	3.6(2.1)	∞*1300*	0/15
MSTART	1.9(0.92)	1.3(0.61)	1.6(0.47)	1.5(0.51)	1.6(0.32)	60(77)	1/15
f_{15}	511	9310	19369	20073	20769	21359	14/15
GLOBAL	6.0(6.9)	∞	∞	∞	∞	∞*2700*	0/15
MSTART	12(9.0)	47(54)	73(85)	71(80)	68(76)	67(73)	1/15
f_{16}	120	612	2663	10449	11644	12095	15/15
GLOBAL	1.4(1.3)	1(0.53)	1(1.1)	3.5(4.1)	6.8(7.6)	6.6(7.2)	0/15
MSTART	0.78(0.79)	2.5(1.8)	7.5(5.9)	19(20)	57(64)	∞*1.0e5*	0/15
f_{17}	5.2	215	899	3669	6351	7934	15/15
GLOBAL	3.5(3.1)	5.0(4.0)	∞	∞	∞	∞*3100*	0/15
MSTART	53(65)	98(109)	734(838)	∞	∞	∞*1.0e5*	0/15
f_{18}	103	378	3968	9280	10905	12469	15/15
GLOBAL	3.9(1.7)	15(14)	14(14)	∞	∞	∞*2600*	0/15
MSTART	41(41)	162(173)	∞	∞	∞	∞*1.0e5*	0/15
f_{19}	1	1	242	1.20e5	1.21e5	1.22e5	15/15
GLOBAL	46(44)	7329(7811)	∞	∞	∞	∞*4300*	0/15
MSTART	19(18)	1651(1624)	179(244)	∞	∞	∞*1.0e5*	0/15
f_{20}	16	851	38111	54470	54861	55313	14/15
GLOBAL	17(4.9)	18(19)	∞	∞	∞	∞*2300*	0/15
MSTART	1.8(1.4)	5.6(8.0)	∞	∞	∞	∞*1.0e5*	0/15
f_{21}	41	1157	1674	1705	1729	1757	14/15
GLOBAL	2.3(2.2)	1.1(0.87)	1(0.85)	1(0.83)	1(0.82)	1(0.81)	14/15
MSTART	4.6(7.0)	2.3(2.5)	2.8(2.6)	2.8(2.5)	2.7(2.5)	2.7(2.5)	15/15
f_{22}	71	386	938	1008	1040	1068	14/15
GLOBAL	3.6(1.7)	1.3(0.90)	1(1.1)	1(1.1)	1(1.0)	1(1.0)	14/15
MSTART	5.3(7.4)	5.9(7.3)	3.1(3.0)	2.9(2.8)	2.9(2.7)	2.9(2.7)	15/15
f_{23}	3.0	518	14249	31654	33030	34256	15/15
GLOBAL	1.6(2.0)	1.0(0.48)	4.8(5.5)	∞	∞	∞*4900*	0/15
MSTART	2.1(2.2)	0.57(0.75)	2.6(3.1)	∞	∞	∞*1.0e5*	0/15
f_{24}	1622	2.16e5	6.36e6	9.62e6	1.28e7	1.28e7	3/15
GLOBAL	4.2(4.7)	∞	∞	∞	∞	∞*6400*	0/15
MSTART	11(12)	∞	∞	∞	∞	∞*1.0e5*	0/15

Table 2: ERT and half-interquantile range (90% – 10%) divided by the best ERT measured during BBOB 2009 for different Δf values for functions f_1–f_{24} in 5-D

Δf	1e+1	1e+0	1e-1	1e-3	1e-5	1e-7	#succ
$\mathbf{f_1}$	43	43	43	43	43	43	15/15
GLOBAL	8.0	8.0	8.0	8.0	8.0	8.0	15/15
MSTART	1	1	1	1	1	1	15/15
$\mathbf{f_2}$	385	386	387	390	391	393	15/15
GLOBAL	18(3.7)	23(3.0)	26(13)	33(14)	51(40)	63(65)	13/15
MSTART	19(4.8)	31(13)	38(27)	69(65)	113(140)	123(139)	12/15
$\mathbf{f_3}$	5066	7626	7635	7643	7646	7651	15/15
GLOBAL	∞	∞	∞	∞	∞	∞*5.0e4*	0/15
MSTART	∞	∞	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_4}$	4722	7628	7666	7700	7758	1.41e5	9/15
GLOBAL	∞	∞	∞	∞	∞	∞*7.8e4*	0/15
MSTART	∞	∞	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_5}$	41	41	41	41	41	41	15/15
GLOBAL	10(0.52)	11(0.78)	11(0.78)	11(0.78)	11(0.78)	11(0.78)	15/15
MSTART	2.2(0.52)	2.9(0.52)	3.1(0.78)	3.1(0.78)	3.1(0.78)	3.1(0.78)	15/15
$\mathbf{f_6}$	1296	2343	3413	5220	6728	8409	15/15
GLOBAL	3.6(1.00)	3.6(0.74)	6.1(3.0)	∞	∞	∞*4.1e4*	0/15
MSTART	3.4(1.3)	4.5(3.1)	6.3(6.0)	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_7}$	1351	4274	9503	16524	16524	16969	15/15
GLOBAL	∞	∞	∞	∞	∞	∞*1.4e4*	0/15
MSTART	∞	∞	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_8}$	2039	3871	4040	4219	4371	4484	15/15
GLOBAL	1.6(0.32)	1.2(0.16)	1.2(0.16)	1.2(0.15)	1.2(0.14)	1.2(0.14)	15/15
MSTART	1.9(0.31)	1.5(0.85)	1.5(0.82)	1.5(0.79)	1.4(0.76)	1.4(0.75)	15/15
$\mathbf{f_9}$	1716	3102	3277	3455	3594	3727	15/15
GLOBAL	1.7(0.28)	1.7(0.89)	1.6(0.84)	1.6(0.79)	1.6(0.77)	1.5(0.74)	15/15
MSTART	2.1(0.26)	1.5(0.09)	1.5(0.09)	1.5(0.09)	1.4(0.08)	1.4(0.08)	15/15
$\mathbf{f_{10}}$	7413	8661	10735	14920	17073	17476	15/15
GLOBAL	1(0.22)	1.1(0.15)	1.1(0.53)	2.0(1.7)	5.9(6.6)	∞*4.1e4*	0/15
MSTART	0.98(0.27)	1.2(0.59)	1.5(0.99)	6.3(7.3)	27(31)	∞*1.1e5*	0/15
$\mathbf{f_{11}}$	1002	2228	6278	9762	12285	14831	15/15
GLOBAL	1.2(0.42)	1.0(0.60)	1(0.84)	∞	∞	∞*2.7e4*	0/15
MSTART	0.87(0.28)	0.78(0.29)	1.6(2.6)	72(79)	∞	∞*1.0e5*	0/15
$\mathbf{f_{12}}$	1042	1938	2740	4140	12407	13827	15/15
GLOBAL	1(0.85)	1(0.88)	1(0.70)	1(0.49)	1.1(1.2)	3.4(3.4)	0/15
MSTART	1.4(0.88)	1.7(0.98)	1.5(0.81)	1.9(1.0)	2.3(3.3)	50(58)	1/15
$\mathbf{f_{13}}$	652	2021	2751	18749	24455	30201	15/15
GLOBAL	2.0(0.34)	1.1(0.08)	1.1(0.04)	4.5(5.0)	∞	∞*1.9e4*	0/15
MSTART	1.7(0.37)	1.0(0.06)	0.99(0.06)	4.4(5.9)	∞	∞*1.0e5*	0/15
$\mathbf{f_{14}}$	75	239	304	932	1648	15661	15/15
GLOBAL	5.0(0.28)	2.2(0.22)	2.1(0.21)	1.1(0.08)	1(0.04)	∞*8500*	0/15
MSTART	2.3(1.3)	1.8(1.1)	1.9(1.0)	1.1(0.46)	1.1(0.35)	∞*1.0e5*	0/15
$\mathbf{f_{15}}$	30378	1.47e5	3.12e5	3.20e5	4.49e5	4.59e5	15/15
GLOBAL	∞	∞	∞	∞	∞	∞*2.4e4*	0/15
MSTART	∞	∞	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_{16}}$	1384	27265	77015	1.88e5	1.98e5	2.20e5	15/15
GLOBAL	1(0.72)	∞	∞	∞	∞	∞*1.2e4*	0/15
MSTART	1.9(2.0)	∞	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_{17}}$	63	1030	4005	30677	56288	80472	15/15
GLOBAL	6.2(1.2)	∞	∞	∞	∞	∞*6.9e4*	0/15
MSTART	386(658)	∞	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_{18}}$	621	3972	19561	67569	1.31e5	1.47e5	15/15
GLOBAL	∞	∞	∞	∞	∞	∞*7.5e4*	0/15
MSTART	∞	∞	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_{19}}$	1	1	3.43e5	6.22e6	6.69e6	6.74e6	15/15
GLOBAL	5601(3531)	∞	∞	∞	∞	∞*5.3e4*	0/15
MSTART	6.75e5(8.07e5)	∞	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_{20}}$	82	46150	3.10e6	5.54e6	5.59e6	5.64e6	14/15
GLOBAL	5.2(0.38)	1.6(1.7)	∞	∞	∞	∞*7.8e4*	0/15
MSTART	2.2(0.38)	5.2(5.8)	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_{21}}$	561	6541	14103	14643	15567	17589	15/15
GLOBAL	1(0.26)	1(1.3)	1(1.2)	1(1.1)	1(1.0)	2.1(2.4)	0/15
MSTART	2.3(4.6)	1.9(2.4)	2.6(3.0)	2.5(2.8)	2.4(2.7)	6.4(7.4)	1/15
$\mathbf{f_{22}}$	467	5580	23491	24948	26847	1.35e5	12/15
GLOBAL	1.1(0.54)	1(1.5)	1(1.2)	1(1.1)	1(0.97)	1.3(1.5)	0/15
MSTART	4.1(5.0)	6.0(9.4)	8.9(10)	8.4(10)	7.9(9.0)	3.6(3.4)	0/15
$\mathbf{f_{23}}$	3.2	1614	67457	4.89e5	8.11e5	8.38e5	15/15
GLOBAL	2.8(2.7)	1(0.93)	∞	∞	∞	∞*9300*	0/15
MSTART	2.0(1.9)	0.62(0.43)	∞	∞	∞	∞*1.0e5*	0/15
$\mathbf{f_{24}}$	1.34e6	7.48e6	5.19e7	5.20e7	5.20e7	5.20e7	3/15
GLOBAL	∞	∞	∞	∞	∞	∞*2.8e4*	0/15
MSTART	∞	∞	∞	∞	∞	∞*1.0e5*	0/15

Table 3: ERT and half-interquantile range (90% – 10%) divided by the best ERT measured during BBOB 2009 for different Δf values for functions f_1–f_{24} in 20-D

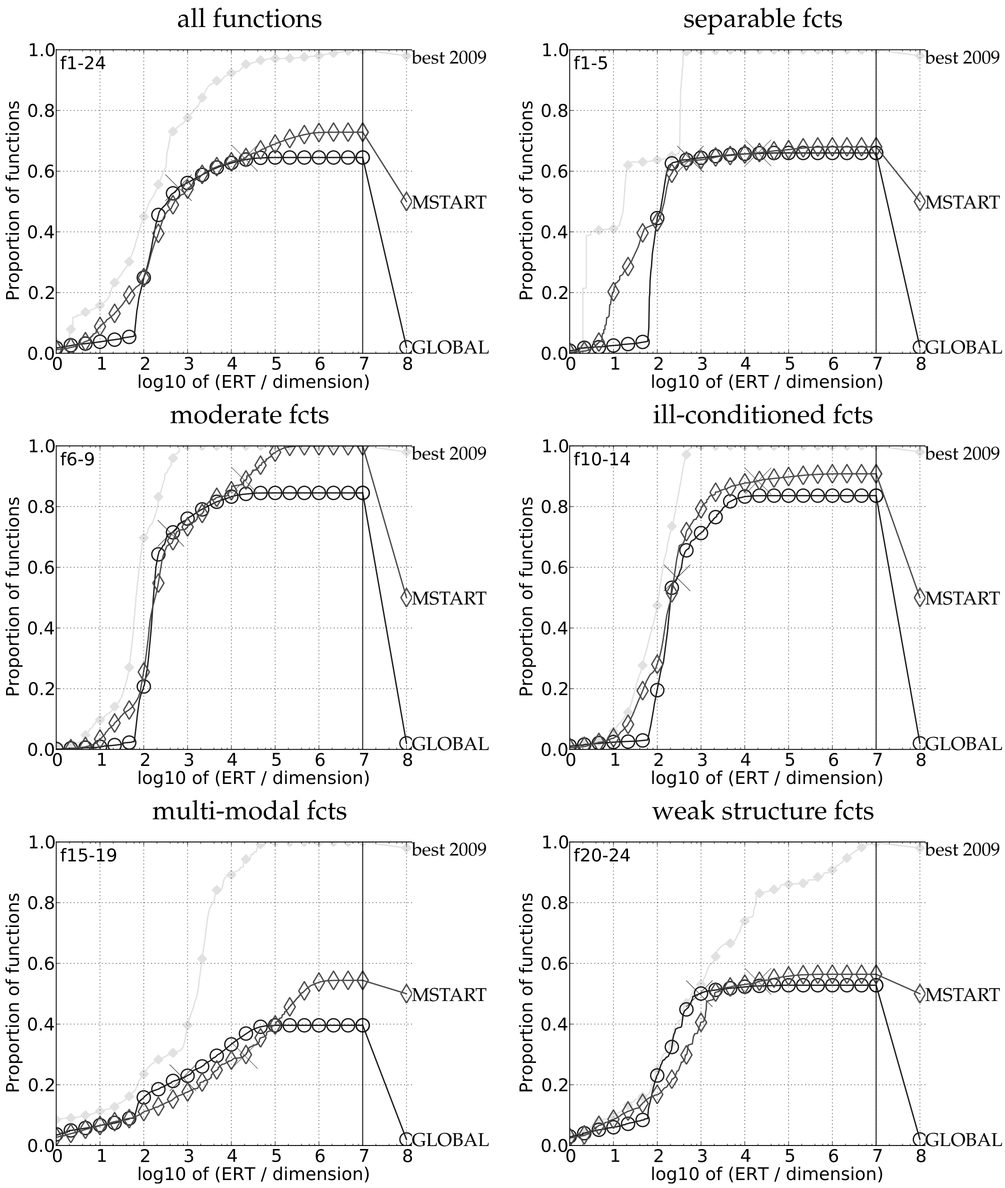


Figure 2: Empirical cumulative distribution of the bootstrapped distribution of ERT over dimension for 50 targets in $10^{[-8..2]}$ for all functions and subgroups in 5-D. The best ever line corresponds to the algorithms from BBOB 2009 with the best ERT for each of the targets considered

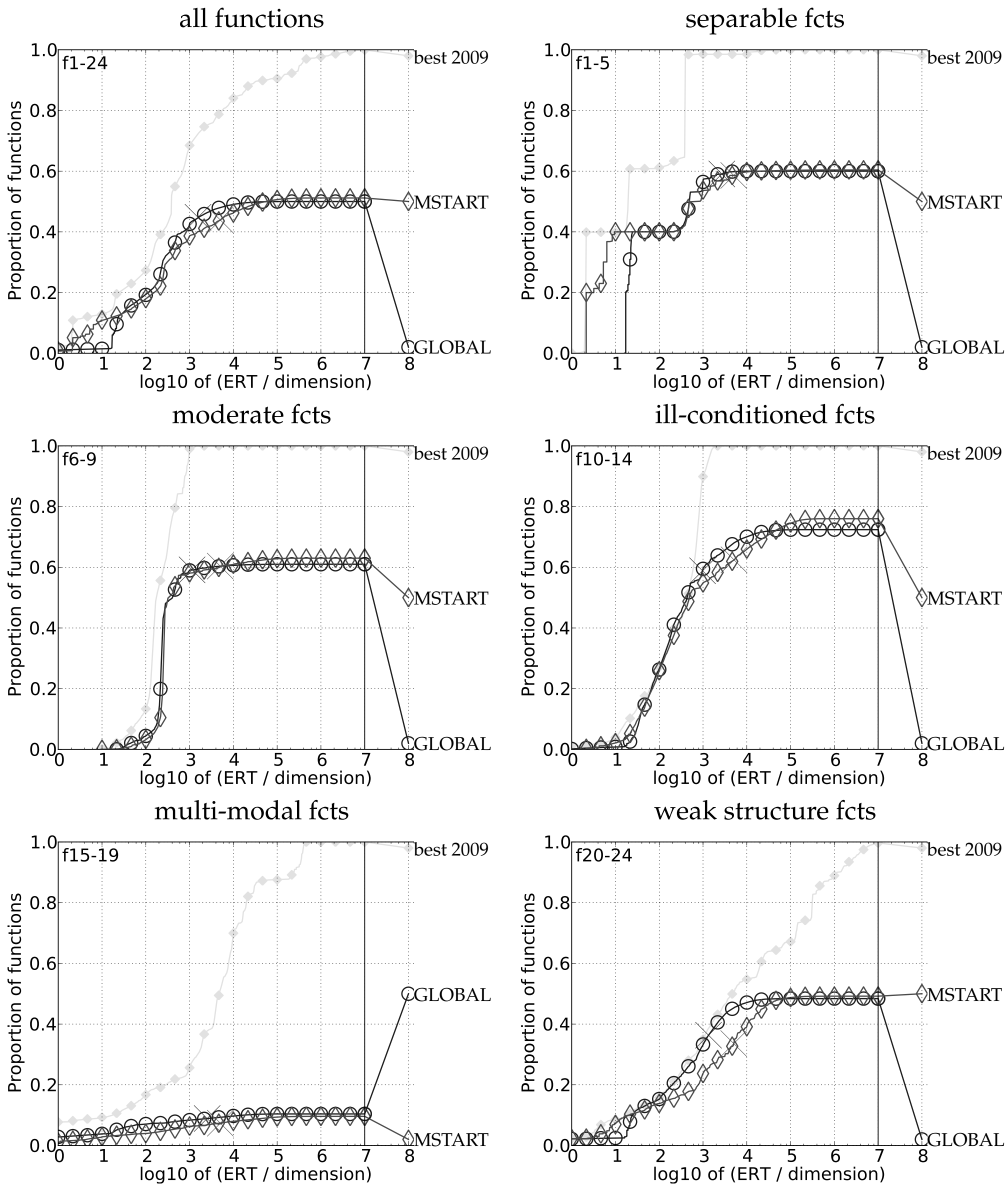


Figure 3: Empirical cumulative distribution of the bootstrapped distribution of ERT over dimension for 50 targets in $10^{[-8..2]}$ for all functions and subgroups in 20-D. The best ever line corresponds to the algorithms from BBOB 2009 with the best ERT for each of the targets considered

Δf	1e+1	1e+0	1e-1	1e-3	1e-5	1e-7	#succ
$\mathbf{f_1}$	11	12	12	12	12	12	15/15
GLOBAL	**0.71**(0.27)$^{\star 2\downarrow}$	1.4(0.49)	**2.0**$^{\star 3}$	**2.6**(0.25)*	**3.4**(0.25)$^{\star 2}$	3.9(0.49)	15/15
GlobalSearch	1.3(0.27)	1.9(0.49)	2.5	3.1(0.25)	3.9(0.25)	4.4(0.49)	15/15
$\mathbf{f_2}$	83	87	88	90	92	94	15/15
GLOBAL	1.8(1.1)	1.9(1.1)	2.0(1.1)	2.2(1.1)	2.8(1.3)	5.0(4.1)	15/15
GlobalSearch	1.9(1.1)	2.0(1.1)	2.0(1.1)	2.2(1.1)	2.9(1.3)	11(19)	15/15
$\mathbf{f_3}$	716	1622	1637	1646	1650	1654	15/15
GLOBAL	7.1(7.7)	276(309)	∞	∞	∞	∞*1.0e5*	0/15
GlobalSearch	6.9(11)	62(65)	∞	∞	∞	∞*1.3e4*	0/15
$\mathbf{f_4}$	809	1633	1688	1817	1886	1903	15/15
GLOBAL	7.5(9.3)	428(460)	∞	∞	∞	∞*1.0e5*	0/15
GlobalSearch	19(22)	∞	∞	∞	∞	∞*1.2e4*	0/15
$\mathbf{f_5}$	10	10	10	10	10	10	15/15
GLOBAL	**2.5**$^{\star 3}$	**4.3**$^{\star 3}$	**5.5**$^{\star 3}$	**6.1**$^{\star 3}$	**6.7**$^{\star 3}$	19(17)	15/15
GlobalSearch	3.1	4.9	6.1	6.7	7.3	133(155)	13/15
$\mathbf{f_6}$	114	214	281	580	1038	1332	15/15
GLOBAL	1.2(0.47)	1.1(0.43)	1.1(0.40)	0.86(0.28)	0.74(0.19)	0.77(0.21)	15/15
GlobalSearch	1.2(0.47)	1.2(0.43)	1.1(0.40)	0.87(0.28)	0.74(0.19)	0.78(0.21)	15/15
$\mathbf{f_7}$	24	324	1171	1572	1572	1597	15/15
GLOBAL	93(73)	1384(1538)	∞	∞	∞	∞*1.0e5*	0/15
GlobalSearch	31(27)	**14**(18)$^{\star 3}$	**35**(41)$^{\star 3}$	∞	∞	∞*6000*	0/15
$\mathbf{f_8}$	73	273	336	391	410	422	15/15
GLOBAL	1.0(0.31)	1.0(1.2)	1.0(0.96)	0.97(0.83)	0.99(0.78)	0.98(0.76)	15/15
GlobalSearch	1.1(0.31)	2.4(6.2)	2.1(5.0)	1.9(4.3)	1.9(4.1)	1.9(4.0)	15/15
$\mathbf{f_9}$	35	127	214	300	335	369	15/15
GLOBAL	**0.94**$^{\star 2}$	**0.64**(0.09)$^{\downarrow 3}$	**0.61**(0.06)$^{\downarrow 4}$	**0.57**(0.04)$^{\downarrow 4}$	**0.56**(0.04)$^{\downarrow 4}$	**0.54**(0.03)$^{\downarrow 4}$	15/15
GlobalSearch	1.1	0.69(0.09)$^{\downarrow 3}$	0.64(0.06)$^{\downarrow 4}$	0.59(0.04)$^{\downarrow 4}$	0.58(0.04)$^{\downarrow 4}$	0.55(0.03)$^{\downarrow 4}$	15/15
$\mathbf{f_{10}}$	349	500	574	626	829	880	15/15
GLOBAL	**0.29**(0.12)$^{\downarrow 3}$	**0.24**(0.13)$^{\downarrow 3}$	**0.24**(0.13)$^{\downarrow 4}$	**0.39**(0.15)$^{\downarrow 2}$	28(61)	136(175)	2/15
GlobalSearch	0.31(0.12)$^{\downarrow 3}$	0.25(0.13)$^{\downarrow 3}$	0.25(0.13)$^{\downarrow 4}$	0.40(0.15)$^{\downarrow 2}$	18(13)	269(348)	1/15
$\mathbf{f_{11}}$	143	202	763	1177	1467	1673	15/15
GLOBAL	**0.24**(0.06)$^{\downarrow 4}$	**0.22**(0.06)$^{\downarrow 4}$	**0.07**(0.02)$^{\downarrow 4}$	2.8(2.9)	75(94)	914(1016)	0/15
GlobalSearch	0.29(0.06)$^{\downarrow 4}$	0.25(0.06)$^{\downarrow 4}$	0.08(0.02)$^{\downarrow 4}$	1.3(2.9)	87(133)	1551(1666)	0/15
$\mathbf{f_{12}}$	108	268	371	461	1303	1494	15/15
GLOBAL	1.3(0.37)	0.93(0.53)	0.89(0.63)	0.92(0.68)	1.6(2.0)	22(34)	6/15
GlobalSearch	1.3(0.37)	0.95(0.53)	0.91(0.63)	0.93(0.68)	9.0(11)	20(29)	2/15
$\mathbf{f_{13}}$	132	195	250	1310	1752	2255	15/15
GLOBAL	**0.74**(0.12)$^{\downarrow 2}$	**0.76**(0.05)$^{\downarrow 3}$	**0.80**(0.06)$^{\downarrow 4}$	3.3(0.31)	34(45)	∞*1.0e5*	0/15
GlobalSearch	0.78(0.12)$^{\downarrow}$	0.79(0.05)$^{\downarrow 3}$	0.82(0.06)$^{\downarrow 4}$	1.7(1.5)	24(27)	∞*2.9e4*	0/15
$\mathbf{f_{14}}$	10	41	58	139	251	476	15/15
GLOBAL	0.68(0.36)	**0.53**(0.22)$^{\downarrow 3}$	**0.64**(0.17)$^{\downarrow 3}$	**0.67**(0.21)$^{\downarrow 2}$	0.73(0.15)	887(1053)	0/15
GlobalSearch	1.1(0.66)	0.68(0.22)$^{\downarrow 2}$	0.74(0.17)$^{\downarrow 2}$	0.71(0.21)$^{\downarrow 2}$	0.75(0.15)	219(238)	0/15
$\mathbf{f_{15}}$	511	9310	19369	20073	20769	21359	14/15
GLOBAL	8.8(7.4)	77(81)	∞	∞	∞	∞*1.0e5*	0/15
GlobalSearch	16(6.6)	28(33)	∞	∞	∞	∞*1.5e4*	0/15
$\mathbf{f_{16}}$	120	612	2663	10449	11644	12095	15/15
GLOBAL	6.8(3.4)	106(104)	∞	∞	∞	∞*1.0e5*	0/15
GlobalSearch	6.7(2.5)	176(222)	256(288)	∞	∞	∞*4.2e4*	0/15
$\mathbf{f_{17}}$	5.2	215	899	3669	6351	7934	15/15
GLOBAL	23(34)	98(133)	∞	∞	∞	∞*1.0e5*	0/15
GlobalSearch	28(49)	156(202)	∞	∞	∞	∞*7.9e4*	0/15
$\mathbf{f_{18}}$	103	378	3968	9280	10905	12469	15/15
GLOBAL	13(20)	1888(2100)	∞	∞	∞	∞*1.0e5*	0/15
GlobalSearch	18(13)	2581(2949)	∞	∞	∞	∞*5.9e4*	0/15
$\mathbf{f_{19}}$	1	1	242	1.20e5	1.21e5	1.22e5	15/15
GLOBAL	1	1	**0.17**(0.04)$^{\downarrow 4}$	∞	∞	∞*1.0e5*	0/15
GlobalSearch	1	1	0.20(0.04)$^{\downarrow 4}$	14(15)	14(15)	13(15)	1/15
$\mathbf{f_{20}}$	16	851	38111	54470	54861	55313	14/15
GLOBAL	**1.4**$^{\star 3}$	4.6(3.8)	2.6(3.0)	1.8(1.9)	1.8(2.2)	1.8(2.1)	9/15
GlobalSearch	1.8	8.0(4.5)	∞	∞	∞	∞*9700*	0/15
$\mathbf{f_{21}}$	41	1157	1674	1705	1729	1757	14/15
GLOBAL	3.4(4.6)	0.87(0.74)	**0.87**(0.67)$^{\star 2}$	**0.88**(0.64)$^{\star 2}$	**0.91**(0.62)$^{\star 2}$	**0.99**(0.63)$^{\star 2}$	15/15
GlobalSearch	4.0(4.8)	9.2(12)	13(16)	13(16)	13(17)	13(15)	7/15
$\mathbf{f_{22}}$	71	386	938	1008	1040	1068	14/15
GLOBAL	2.5(2.7)	1.6(2.4)	1.5(1.7)	1.4(1.5)	1.5(1.5)	1.6(1.5)	15/15
GlobalSearch	4.6(4.7)	14(18)	8.9(11)	8.9(11)	8.7(10)	8.6(10)	9/15
$\mathbf{f_{23}}$	3.0	518	14249	31654	33030	34256	15/15
GLOBAL	6.9(6.0)	2.6(2.5)	2.1(2.1)	13(16)	∞	∞*1.0e5*	0/15
GlobalSearch	8.5(7.0)	4.1(3.7)	2.1(2.0)	12(11)	∞	∞*4.7e4*	0/15
$\mathbf{f_{24}}$	1622	2.16e5	6.36e6	9.62e6	1.28e7	1.28e7	3/15
GLOBAL	3.2(3.4)	6.9(7.2)	∞	∞	∞	∞*1.0e5*	0/15
GlobalSearch	3.7(2.4)	∞	∞	∞	∞	∞*1.5e4*	0/15

Table 4: ERT and half-interquantile range (90% – 10%) divided by the best ERT measured during BBOB 2009 for different Δf values for functions f_1–f_{24} in 5-D

Δf	1e+1	1e+0	1e-1	1e-3	1e-5	1e-7	#succ
$\mathbf{f_1}$	43	43	43	43	43	43	15/15
GLOBAL	**0.77**(0.24)*	1.7(0.49)	**1.9**(0.23)$^{\star 3}$	**2.8**(0.23)$^{\star 3}$	3.7(0.48)	4.7(0.49)	15/15
GlobalSearch	1.3(0.24)	2.1(0.49)	2.3(0.23)	3.3(0.23)	4.2(0.48)	5.1(0.49)	15/15
$\mathbf{f_2}$	385	386	387	390	391	393	15/15
GLOBAL	4.8(1.4)	5.3(1.5)	5.7(1.7)	6.6(1.8)	10(2.7)	13(7.7)	15/15
GlobalSearch	4.9(1.4)	5.4(1.5)	5.8(1.7)	6.6(1.8)	12(2.7)	16(20)	15/15
$\mathbf{f_3}$	5066	7626	7635	7643	7646	7651	15/15
GLOBAL	∞	∞	∞	∞	∞	∞*4.0e5*	0/15
GlobalSearch	∞	∞	∞	∞	∞	∞*2.4e5*	0/15
$\mathbf{f_4}$	4722	7628	7666	7700	7758	1.41e5	9/15
GLOBAL	∞	∞	∞	∞	∞	∞*4.0e5*	0/15
GlobalSearch	∞	∞	∞	∞	∞	∞*2.4e5*	0/15
$\mathbf{f_5}$	41	41	41	41	41	41	15/15
GLOBAL	**3.6**$^{\star 3}$	**5.2**$^{\star 3}$	**5.7**$^{\star 3}$	**6.7**$^{\star 3}$	**7.3**$^{\star 3}$	**110**(74)$^{\star 3}$	4/15
GlobalSearch	4.2	5.7	6.2	7.3	7.8	2320(4052)	6/15
$\mathbf{f_6}$	1296	2343	3413	5220	6728	8409	15/15
GLOBAL	3.9(2.3)	2.7(2.1)	2.4(1.6)	2.4(1.3)	**2.5**(1.1)$^{\star 2}$	**2.6**(0.97)$^{\star 3}$	14/15
GlobalSearch	2.9(2.3)	2.6(4.5)	2.2(3.1)	3.0(2.0)	29(38)	∞*2.5e5*	0/15
$\mathbf{f_7}$	1351	4274	9503	16524	16524	16969	15/15
GLOBAL	∞	∞	∞	∞	∞	∞*4.0e5*	0/15
GlobalSearch	∞	∞	∞	∞	∞	∞*3.3e4*	0/15
$\mathbf{f_8}$	2039	3871	4040	4219	4371	4484	15/15
GLOBAL	0.84(0.18)	0.76(0.39)	0.77(0.38)	0.77(0.36)	0.77(0.35)	**0.76**(0.34)$^{\downarrow}$	15/15
GlobalSearch	0.85(0.18)	1.1(1.9)	1.1(1.8)	1.1(1.7)	1.1(1.7)	1.1(1.6)	15/15
$\mathbf{f_9}$	1716	3102	3277	3455	3594	3727	15/15
GLOBAL	**0.17**(0.01)$^{\downarrow 4}$	**0.34**(0.02)$^{\downarrow 4}$	**0.38**(0.02)$^{\downarrow 4}$	**0.40**(0.02)$^{\downarrow 4}$	**0.40**(0.02)$^{\downarrow 4}$	**0.40**(0.02)$^{\downarrow 4}$	15/15
GlobalSearch	0.19(0.01)$^{\downarrow 4}$	0.35(0.02)$^{\downarrow 4}$	0.38(0.02)$^{\downarrow 4}$	0.40(0.02)$^{\downarrow 4}$	0.41(0.02)$^{\downarrow 4}$	0.40(0.02)$^{\downarrow 4}$	15/15
$\mathbf{f_{10}}$	7413	8661	10735	14920	17073	17476	15/15
GLOBAL	**0.17**(0.07)$^{\downarrow 4}$	**0.16**(0.06)$^{\downarrow 4}$	**0.14**(0.05)$^{\downarrow 4}$	**0.12**(0.04)$^{\downarrow 4}$	50(61)	∞*4.1e5*	0/15
GlobalSearch	0.17(0.07)$^{\downarrow 4}$	0.16(0.06)$^{\downarrow 4}$	0.14(0.05)$^{\downarrow 4}$	0.12(0.04)$^{\downarrow 4}$	51(64)	∞*3.1e5*	0/15
$\mathbf{f_{11}}$	1002	2228	6278	9762	12285	14831	15/15
GLOBAL	**0.16**(0.04)$^{\downarrow 4}$	**0.10**(0.02)$^{\downarrow 4}$	**0.04**(0.01)$^{\downarrow 4}$	0.58(1.4)	470(530)	∞*4.0e5*	0/15
GlobalSearch	0.18(0.04)$^{\downarrow 4}$	0.10(0.02)$^{\downarrow 4}$	0.05(0.01)$^{\downarrow 4}$	0.48(1.1)	380(399)	∞*3.2e5*	0/15
$\mathbf{f_{12}}$	1042	1938	2740	4140	12407	13827	15/15
GLOBAL	0.80(0.57)	0.91(0.55)	0.83(0.51)	0.73(0.29)	0.67(0.74)	21(29)	3/15
GlobalSearch	0.82(0.57)	0.92(0.55)	0.84(0.51)	0.74(0.29)	6.0(13)	25(37)	3/15
$\mathbf{f_{13}}$	652	2021	2751	18749	24455	30201	15/15
GLOBAL	1.1(0.20)	**0.59**(0.17)$^{\downarrow 2}$	0.89(0.74)	**0.46**(0.55)$^{\downarrow}$	∞	∞*4.0e5*	0/15
GlobalSearch	1.2(0.20)	0.93(0.17)$^{\downarrow 2}$	1.9(2.7)	1.7(2.3)	∞	∞*3.1e5*	0/15
$\mathbf{f_{14}}$	75	239	304	932	1648	15661	15/15
GLOBAL	0.74(0.27)	**0.49**(0.09)$^{\downarrow 3}$	**0.65**(0.14)$^{\downarrow 2}$	**0.65**(0.06)$^{\downarrow 4}$	**0.69**(0.14)$^{\downarrow 4}$	∞*4.0e5*	0/15
GlobalSearch	1.0(0.27)	0.58(0.09)$^{\downarrow 2}$	0.72(0.14)$^{\downarrow}$	0.68(0.06)$^{\downarrow 4}$	0.71(0.14)$^{\downarrow 4}$	∞*3.4e5*	0/15
$\mathbf{f_{15}}$	30378	1.47e5	3.12e5	3.20e5	4.49e5	4.59e5	15/15
GLOBAL	∞	∞	∞	∞	∞	∞*4.0e5*	0/15
GlobalSearch	∞	∞	∞	∞	∞	∞*2.5e5*	0/15
$\mathbf{f_{16}}$	1384	27265	77015	1.88e5	1.98e5	2.20e5	15/15
GLOBAL	508(484)	∞	∞	∞	∞	∞*4.0e5*	0/15
GlobalSearch	394(428)	∞	∞	∞	∞	∞*2.2e5*	0/15
$\mathbf{f_{17}}$	63	1030	4005	30677	56288	80472	15/15
GLOBAL	20(32)	∞	∞	∞	∞	∞*4.0e5*	0/15
GlobalSearch	23(35)	∞	∞	∞	∞	∞*3.2e5*	0/15
$\mathbf{f_{18}}$	621	3972	19561	67569	1.31e5	1.47e5	15/15
GLOBAL	∞	∞	∞	∞	∞	∞*4.0e5*	0/15
GlobalSearch	2180(2247)	∞	∞	∞	∞	∞*3.1e5*	0/15
$\mathbf{f_{19}}$	1	1	3.43e5	6.22e6	6.69e6	6.74e6	15/15
GLOBAL	1	1	**0.00**$^{\star 3 \downarrow 4}$	∞	∞	∞*4.0e5*	0/15
GlobalSearch	1	1	0.00$^{\downarrow 4}$	∞	∞	∞*2.8e5*	0/15
$\mathbf{f_{20}}$	82	46150	3.10e6	5.54e6	5.59e6	5.64e6	14/15
GLOBAL	**1.4**$^{\star 3}$	**10**(8.5)*	∞	∞	∞	∞*4.0e5*	0/15
GlobalSearch	1.7	∞	∞	∞	∞	∞*1.8e5*	0/15
$\mathbf{f_{21}}$	561	6541	14103	14643	15567	17589	15/15
GLOBAL	1.1(1.6)	0.87(1.3)	**0.88**(0.82)*	**0.86**(0.79)*	**0.84**(0.75)*	**0.81**(0.66)*	8/15
GlobalSearch	7.1(12)	63(82)	53(63)	51(65)	48(59)	43(54)	2/15
$\mathbf{f_{22}}$	467	5580	23491	24948	26847	1.35e5	12/15
GLOBAL	3.3(3.6)	4.2(4.6)	4.7(7.7)	4.4(7.3)	4.2(6.8)	0.97(1.3)	4/15
GlobalSearch	106(312)	105(129)	48(59)	45(55)	42(49)	8.4(10)	1/15
$\mathbf{f_{23}}$	3.2	1614	67457	4.89e5	8.11e5	8.38e5	15/15
GLOBAL	9.0(10)	4.8(6.5)	27(30)	∞	∞	∞*4.0e5*	0/15
GlobalSearch	13(14)	6.6(6.1)	16(18)	∞	∞	∞*2.4e5*	0/15
$\mathbf{f_{24}}$	1.34e6	7.48e6	5.19e7	5.20e7	5.20e7	5.20e7	3/15
GLOBAL	∞	∞	∞	∞	∞	∞*4.0e5*	0/15
GlobalSearch	∞	∞	∞	∞	∞	∞*2.6e5*	0/15

Table 5: ERT and half-interquantile range (90% – 10%) divided by the best ERT measured during BBOB 2009 for different Δf values for functions f_1–f_{24} in 20-D

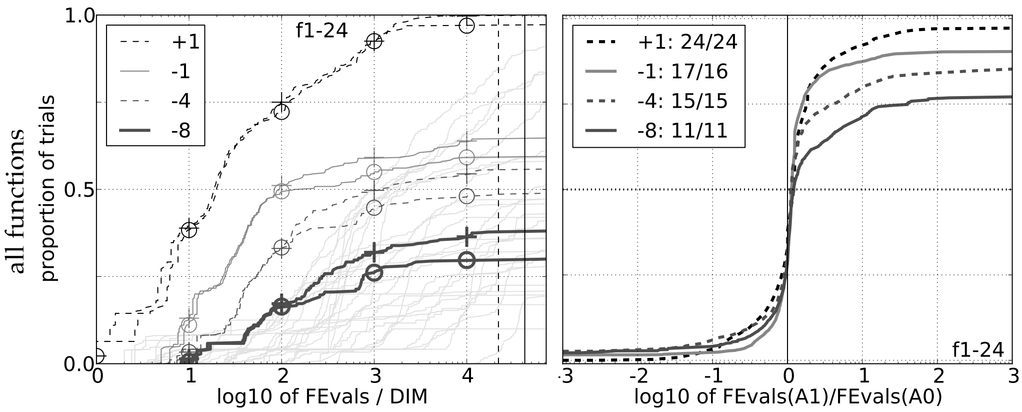


Figure 4: Noiseless functions 5-D. Left: Empirical Cumulative Distribution Function (ECDF) of the running time (number of function evaluations) for GlobalSearch (○) and GLOBAL (+), divided by search space dimension D, to fall below $f_{\rm opt} + \Delta f$ with $\Delta f = 10^k$ where k is the value in the legend. The vertical black lines indicate the maximum number of function evaluations. Light brown lines in the background show ECDFs for target value 10^{-8} of all algorithms benchmarked during BBOB 2009. Right subplots: ECDF of ERT of GlobalSearch over ERT of GLOBAL for different Δf

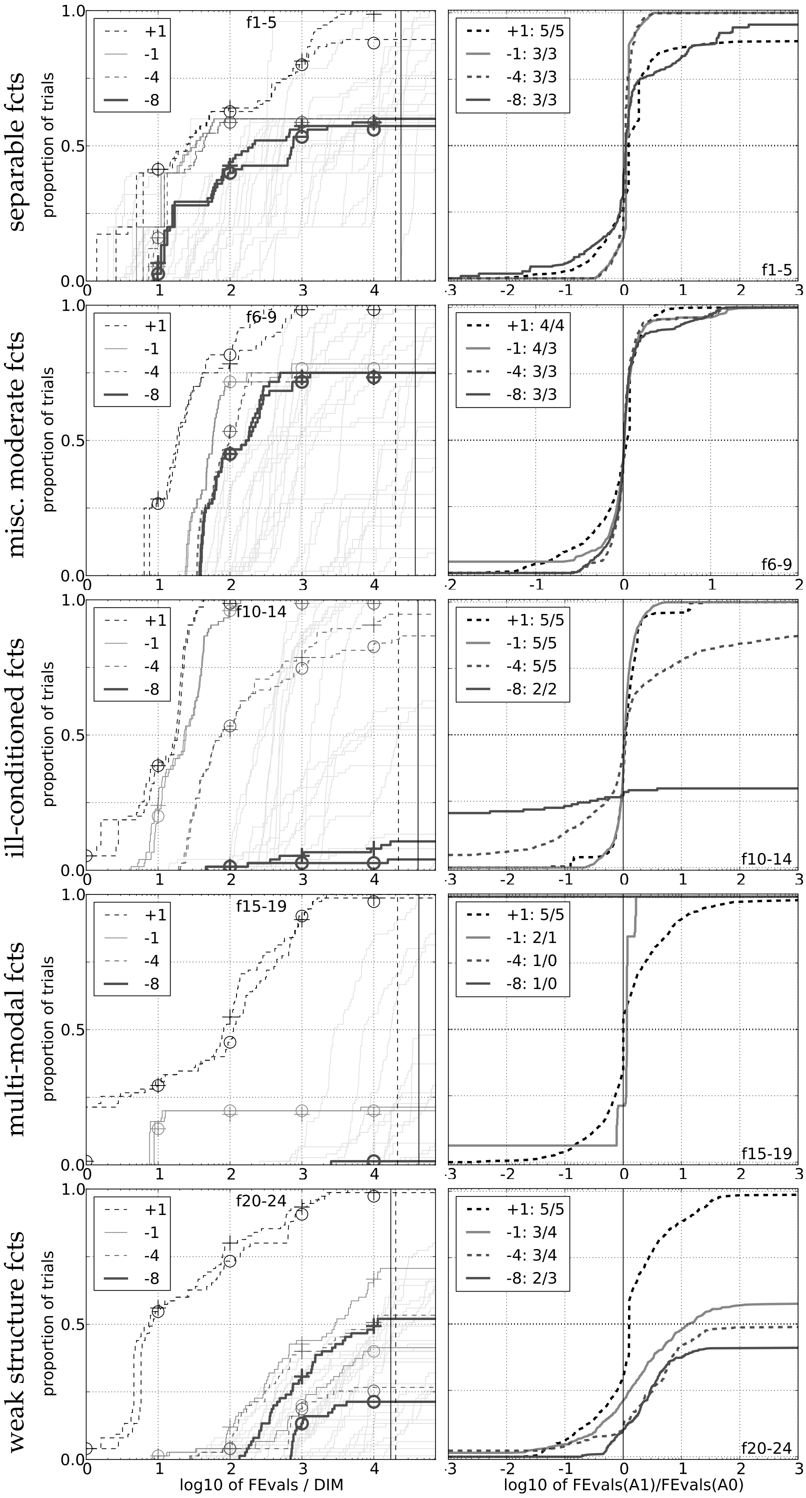


Figure 5: Subgroups of functions 5-D. See caption of Figure 4

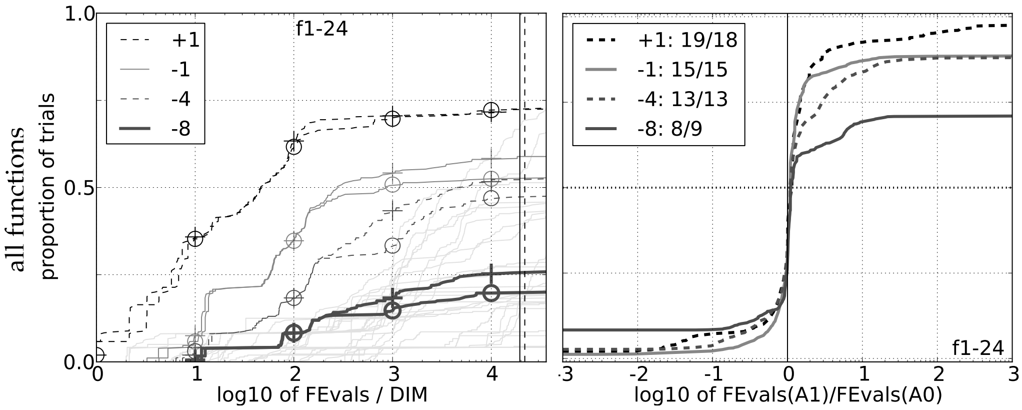


Figure 6: Noiseless functions 20-D. See caption of Figure 4

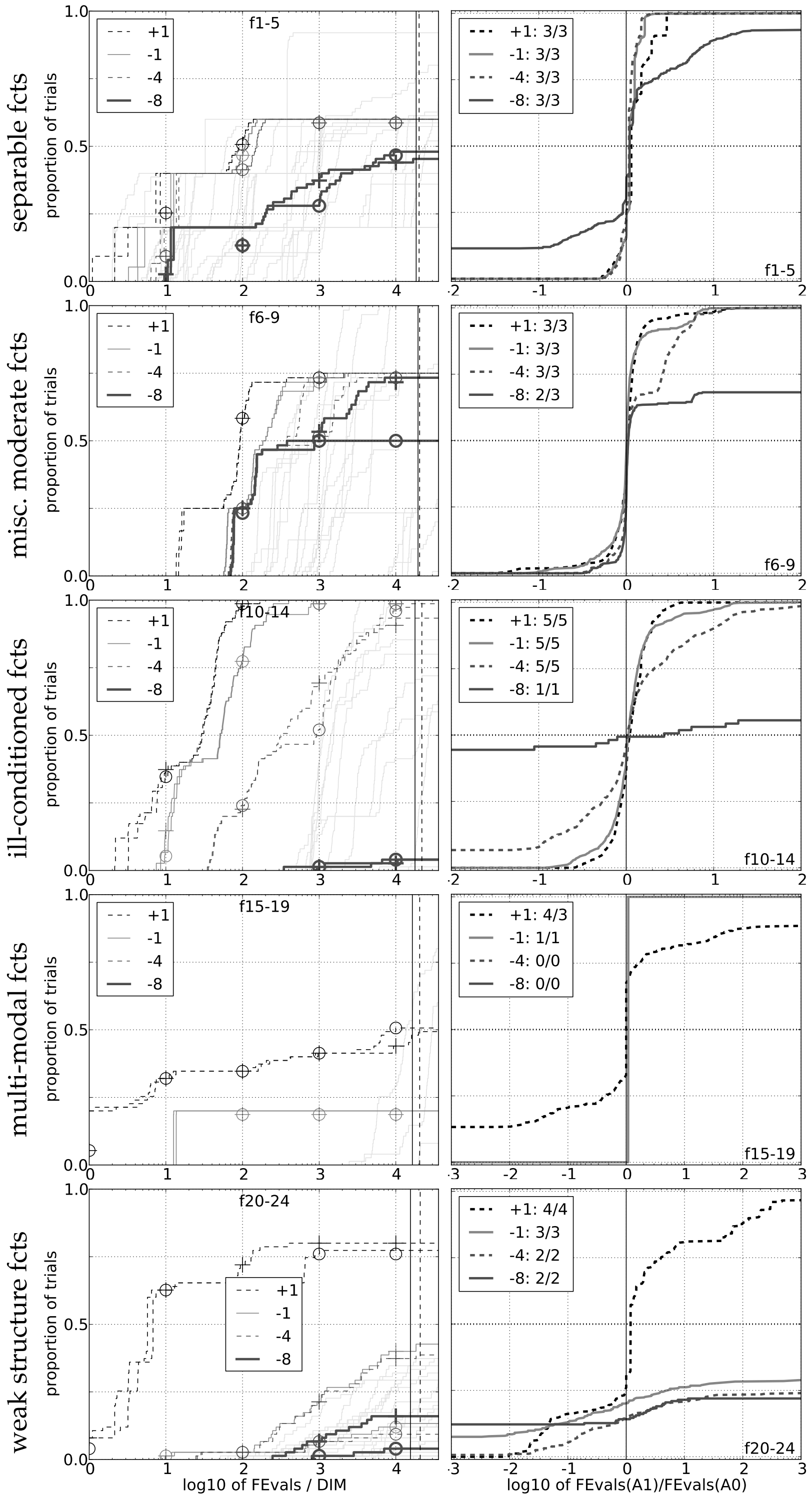
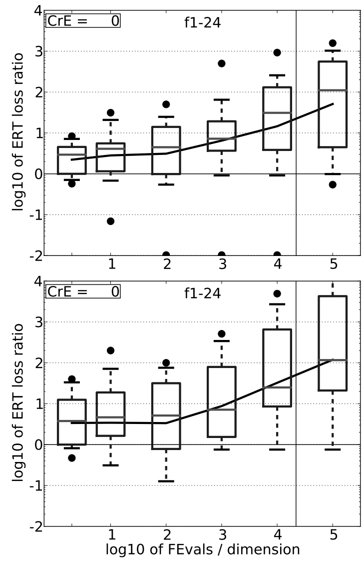


Figure 7: Subgroups of functions 20-D. See caption of Figure 4



	f_1–f_{24} in 5-D, maxFE/D=21991					
#FEs/D	best	10%	25%	**med**	75%	90%
2	0.57	0.70	1.0	2.7	4.5	7.1
10	0.07	0.63	1.0	4.1	5.4	21
100	0.01	0.51	0.92	4.3	12	25
1e3	0.01	0.88	3.5	6.6	17	80
1e4	0.01	0.88	3.7	27	95	2.9e2
1e5	0.54	0.98	4.3	1.1e2	5.1e2	1.1e3
RL_{US}/D	2e4	2e4	2e4	2e4	2e4	2e4

	f_1–f_{24} in 20-D, maxFE/D=21857					
#FEs/D	best	10%	25%	**med**	75%	90%
2	0.47	0.81	1.0	2.9	11	34
10	0.00	0.29	1.2	4.5	14	81
100	0.00	0.12	0.77	4.7	31	77
1e3	0.00	0.72	1.4	6.7	74	3.6e2
1e4	0.00	0.72	6.8	22	5.1e2	2.8e3
1e5	0.00	0.72	15	1.2e2	3.3e3	1.9e4
RL_{US}/D	2e4	2e4	2e4	2e4	2e4	2e4

Figure 8: ERT loss ratio for the GLOBAL method. Left: plotted versus given budget $\mathrm{FEvals} = \#\mathrm{FEs}$ in log-log display. Box-Whisker plot shows 25-75%-ile (box) with median, 10-90%-ile (caps), and minimum and maximum ERT loss ratio (points). The black line is the geometric mean. The vertical line gives the maximal number of function evaluations. Right: tabulated ERT loss ratios in 5-D (top table) and 20-D (bottom table). maxFE/D gives the maximum number of function evaluations divided by the dimension. RL_{US}/D gives the median number of function evaluations for unsuccessful trials.

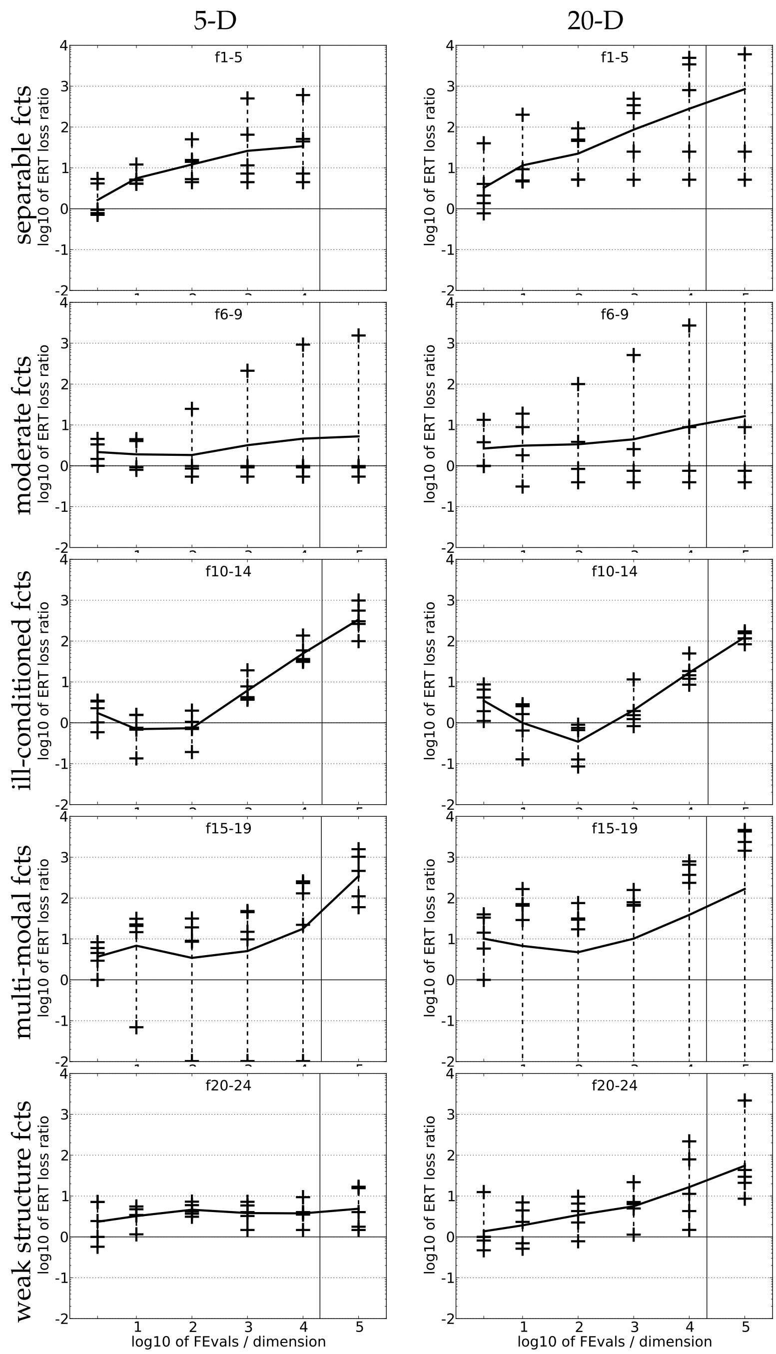
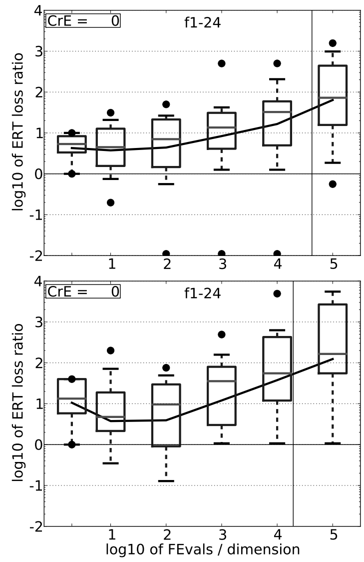


Figure 9: ERT loss ratio versus given budget FEvals divided by dimension in log-log display for the GLOBAL method. Crosses give the single values on the indicated functions, the line is the geometric mean. The vertical line gives the maximal number of function evaluations in the respective function subgroup.



		***f*1–*f*24 in 5-D**, maxFE/D=42330				
#FEs/D	best	10%	25%	**med**	75%	90%
2	1.0	1.0	3.1	5.0	7.7	10
10	0.20	0.70	1.5	4.4	12	21
100	0.01	0.52	1.3	6.5	21	27
1e3	0.01	1.2	3.9	11	29	44
1e4	0.01	1.2	4.5	31	55	2.1e2
1e5	0.56	1.8	15	66	4.3e2	9.9e2
$\mathrm{RL_{US}}$/D	1e3	1e3	2e3	4e3	1e4	2e4

		***f*1–*f*24 in 20-D**, maxFE/D=19469				
#FEs/D	best	10%	25%	**med**	75%	90%
2	1.0	1.0	5.0	13	40	40
10	0.00	0.33	1.5	4.7	14	81
100	0.00	0.12	0.88	7.6	26	49
1e3	0.00	1.0	2.5	24	80	1.8e2
1e4	0.00	1.0	10	53	4.0e2	7.5e2
1e5	0.00	1.0	43	1.4e2	2.1e3	5.8e3
$\mathrm{RL_{US}}$/D	1e3	9e3	1e4	1e4	2e4	2e4

Figure 10: ERT loss ratio for the GlobalSearch solver. Left: plotted versus given budget FEvals = #FEs in log-log display. Box-Whisker plot shows 25-75%-ile (box) with median, 10-90%-ile (caps), and minimum and maximum ERT loss ratio (points). The black line is the geometric mean. The vertical line gives the maximal number of function evaluations. Right: tabulated ERT loss ratios in 5-D (top table) and 20-D (bottom table). maxFE/D gives the maximum number of function evaluations divided by the dimension. $\mathrm{RL_{US}}$/D gives the median number of function evaluations for unsuccessful trials.

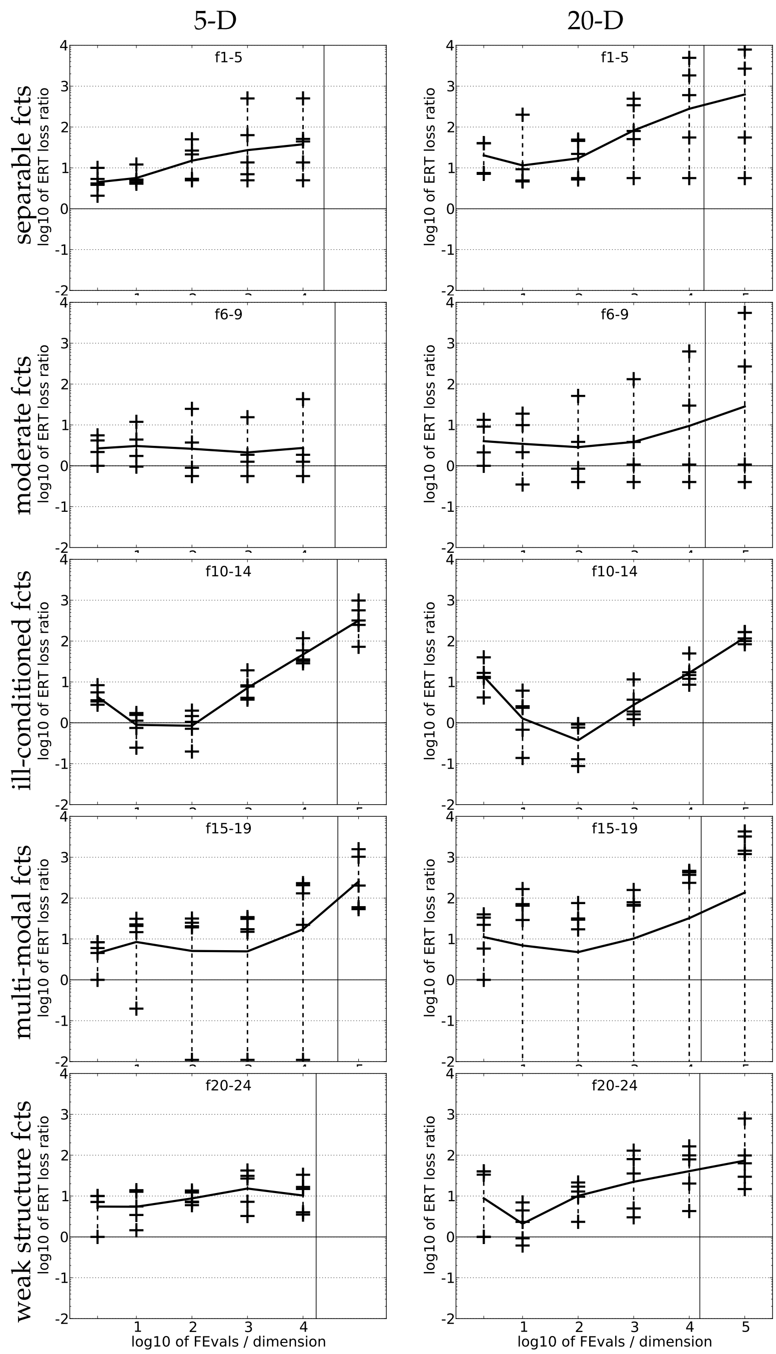


Figure 11: ERT loss ratio versus given budget FEvals divided by dimension in log-log display for the GlobalSearch solver. Crosses give the single values on the indicated functions, the line is the geometric mean. The vertical line gives the maximal number of function evaluations in the respective function subgroup.

References

Betrò, B. and Schoen, F. (1992). Optimal and sub-optimal stopping rules for the multistart algorithm in global optimization. *Mathematical Programming*, 57:445–458.

Boender, C., Rinnooy Kan, A., Timmer, G., and Stougie, L. (1982a). A stochastic method for global optimization. *Mathematical Programming*, 22:125–140.

Boender, C. G. E. and Rinnooy Kan, A. H. G. (1987). Bayesian stopping rules for multistart global optimization methods. *Mathematical Programming*, 37:59–80.

Boender, C. G. E. and Rinnooy Kan, A. H. G. (1991). On when to stop sampling for the maximum. *Journal of Global Optimization*, 1:331–340.

Boender, C. G. E., Rinnooy Kan, A. H. G., Timmer, G. T., and Stougie, L. (1982b). A stochastic method for global optimization. *Mathematical Programming*, 22:125–140.

Boender, C. G. E. and Zieliński, R. (1982). A sequential bayesian approach to estimating the dimension of a multinominal distribution. In *Sequential Methods in Statistics. Banach Center Publications*, volume 16. PWN–Polish Scientific Publisher, Warsaw.

Csendes, T. (1988). Nonlinear parameter estimation by global optimization—efficiency and reliability. *Acta Cybernetica*, 8(4):361–370.

Csendes, T., Pál, L., Sendin, J. O. H., and Banga, J. R. (2008). The GLOBAL Optimization Method Revisited. *Optimization Letters*, 2:445–454.

Davidon, W. (1959). Variable metric method for minimization. Technical Report ANL5990 (revised), Argonne National Laboratory, Argonne, Il.

Finck, S., Hansen, N., Ros, R., and Auger, A. (2009a). Real-parameter black-box optimization benchmarking 2009: Presentation of the noiseless functions. Technical Report 2009/20, Research Center PPE.

Finck, S., Hansen, N., Ros, R., and Auger, A. (2009b). Real-parameter black-box optimization benchmarking 2009: Presentation of the noiseless functions. Technical Report 2009/20, Research Center PPE. Updated February 2010.

Guss, C., Boender, C. G. E., and Romeijn, H. E. (1995). Stochastic methods. In Horst, R. and Pardalos, P., editors, *Handbook of Global Optimization*, pages 829–869. Kluwer Academic Publishers, Dordrecht.

Hansen, N., Auger, A., Finck, S., and Ros, R. (2009a). Real-parameter black-box optimization benchmarking 2009: Experimental setup. Technical Report RR-6828, INRIA.

Hansen, N., Auger, A., Finck, S., and Ros, R. (2010a). Real-parameter black-box optimization benchmarking 2010: Experimental setup. Technical Report RR-7215, INRIA.

Hansen, N., Auger, A., Ros, R., Finck, S., and Pošík, P. (2010b). Comparing results of 31 algorithms from the black-box optimization benchmarking bbob-2009. In *GECCO '10: Proceedings of the 12th annual conference comp on Genetic and evolutionary computation*, pages 1689–1696, New York, NY, USA. ACM.

Hansen, N., Finck, S., Ros, R., and Auger, A. (2009b). Real-parameter black-box optimization benchmarking 2009: Noiseless functions definitions. Technical Report RR-6829, INRIA.

Hansen, N., Finck, S., Ros, R., and Auger, A. (2009c). Real-parameter black-box optimization benchmarking 2009: Noiseless functions definitions. Technical Report RR-6829, INRIA. Updated February 2010.

Hoos, H. H. and Stützle, T. (1998). Evaluating Las Vegas algorithms—pitfalls and remedies. In *Proceedings of the Fourteenth Conference on Uncertainty in Artificial Intelligence (UAI-98)*, pages 238–245.

Huyer, W. and Neumaier, A. (1999). Global optimization by multilevel coordinate search. *J. of Global Optimization*, 14(4):331–355.

Järvi, T. (1973). A random search optimizer with an application to a max-min problem. *Publications of the Institute for Applied Mathematics*, (3). University of Turku, Finland.

Kelley, C. T. (1999). *Iterative Methods for Optimization*. SIAM, Philadelphia.

Locatelli, M. and Schoen, F. (1999). Random linkage: a family of acceptance/rejection algorithms for global optimization. *Mathematical Programming*, 2:379–396.

Moles, C., Gutierrez, G., Alonso, A., and Banga, J. (2003). Integrated Process Design and Control Via Global Optimization – A Wastewater Treatment Plant Case Study. *Chemical Engineering Research and Design*, 81:507–517.

Mongeau, M., Karsenty, H., Rouzé, V., and Hiriart-Urruty, J. (2000). Comparison of public-domain software for black box global optimization. *Optimization Methods and Software*, 13:203–226.

Nelder, J. and Mead, R. (1965). The downhill simplex method. *Computer Journal*, 7:308–313.

Pál, L., Csendes, T., Markót, M. C., and Neumaier, A. (2009). BBO-benchmarking of the GLOBAL method for the noiseless function testbed. `http://www.mat.univie.ac.at/~neum/papers.html`. P. 986.

Powell, M. (1986). How bad are the BFGS and DFP methods when the objective function is quadratic? *Mathematical Programming*, 34:34–47.

Powell, M. J. D. (2006). The NEWUOA software for unconstrained optimization without derivatives. *Large Scale Nonlinear Optimization*, pages 255–297.

Price, K. (1997). Differential evolution vs. the functions of the second ICEO. In *Proceedings of the IEEE International Congress on Evolutionary Computation*, pages 153–157.

Rinnooy Kan, A. H. G. and Timmer, G. T. (1987a). Stochastic global optimization methods Part I: Clustering methods. *Mathematical Programming*, 39:27–56.

Rinnooy Kan, A. H. G. and Timmer, G. T. (1987b). Stochastic global optimization methods Part I: Clustering methods. *Mathematical Programming*, 39:57–78.

Rios, L. M. and Sahinidis, N. V. (2010). Derivative-free optimization: a review of algorithms and comparison of software implementations. Technical report, Department of Chemical Engineering, Carnegie Mellon University, Pittsburgh, PA. http://thales.cheme.cmu.edu/dfo.

Törn, A. A. (1978). A search clustering approach to global optimization. In Dixon, L. and Szegő, G., editors, *Towards Global Optimization* 2. North-Holland, Amsterdam, The Netherlands.

Ugray, Z., Lasdon, L., Plummer, J., Glover, R., Kelly, J., and Marti, R. (2007). Scatter search and local nlp solvers: A multistart framework for global optimization. *INFORMS Journal on Computing*, 19(3):328–340.

Zieliński, R. (1981). A stochastic estimate of the structure of multi-extremal problems. *Mathematical Programming*, 21:348–356.